Outer Bound of the Capacity Region for Identification via Multiple Access Channels

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***Abstract*—In this paper we consider the identification (ID) via multiple access channels (MACs). In the general MAC the ID capacity region includes the ordinary transmission (TR) capacity region. In this paper we discuss the converse coding theorem. We estimate two types of error probabilities of identification for rates outside capacity region, deriving some function which serves as a lower bound of the sum of two error probabilities of identification. This function has a property that it tends to zero as $n \to \infty$ for noisy channels satisfying the strong converse property. Using this property, we establish that the transmission capacity region is equal to the ID capacity for the MAC satisfying the strong converse property. To derive the result we introduce a new resolvability problem on the output from the MAC. We further develop a new method of converting the direct coding theorem for the above MAC resolvability problem into the converse coding theorem for the ID via MACs.**

I. Introduction

In 1989, Ahlswede and Dueck [1],[2], proposed a new framework of communication system using noisy channels. Their proposed framework called the identification via channels (or briefly say the ID channel) has opened a new and fertile area in the Shannon theory. After their pioneering work, the ID channel coding problem has intensively been studied from both theoretical and practical point of view ([3]-[13]). Identification via multi-way channels is an interesting problem. This problem was studied by [6], [8], [14] and [15]. In spite of its theoretical interest and practical importance, the number of works on this theme seems to be relatively few.

In this paper we deal with the identification via multiple access channels (MACs) for general noisy channels with two inputs and one output finite sets and channel transition probabilities that may be arbitrary for every block length n. Steinberg [8], and the author studied the identification(ID) capacity region for general MACs. However, these works have a common gap in the proofs of the converse coding theorems. This gap was pointed out by Hayashi [12] and is not resolved yet.

According to Steinberg [8], by a similar argument to the case of single user channels we can show that the ID capacity region contains the transmission(TR) capacity region for the general MAC. He studied the converse coding theorem by using a lemma used to prove the converse coding theorem for the ID via single user channels. In this paper we focus on our attention to the converse coding theorem and study it by an approach different from that of Steinberg. We estimate two types of error probabilities of identification for rates outside capacity region, deriving some function which serves as a lower bound of the sum of two error probabilities of identification. This function has a property that it tends to zero as $n \to \infty$ for noisy channels satisfying the strong converse property. Using this property, we establish that the transmission capacity region is equal to the ID capacity for the MAC satisfying the strong converse property.

To derive the converse coding theorem for the ID channel Han and Verdú [4] introduced an approximation problem of output distributions from single user channels. They call this problem channel resolvability problem. They first proved a direct coding theorem for the channel coding theorem and next proved a converse coding theorem for the ID channel by converting the direct coding theorem for the channel resolvability problem into the converse coding theorem for the ID channel. To prove the converse coding theorem for the ID via MACs, we formulate a new approximation problem of output distributions from MACs. This problem is regard as a MAC resolvability problem. A similar resolvability problem using MACs was studied by Steinberg [17]. Our problem is some variant of his problem. We first establish a stronger result on the direct coding theorem for this problem by deriving an upper bound for the approximation error of channel outputs to tend to zero as n goes to infinity. Next, we prove the converse coding theorem by converting the direct coding theorem for the MAC resolvability problem into the converse coding theorem for the ID via MACs.

II. Identification via Multiple Access Channels

Let $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ be finite sets. Let $\mathcal{P}(\mathcal{X}^n)$ and $\mathcal{P}(\mathcal{Y}^n)$ be sets of probability distributions on $\mathcal{X}^n$ and $\mathcal{Y}^n$, respectively. A source $\boldsymbol{X}$ with alphabet $\mathcal{X}$ is the sequence $\{P_X^n : P_X^n \in \mathcal{P}(\mathcal{X}^n)\}_{n=1}^{\infty}$ and a source $\boldsymbol{Y}$ with alphabet $\mathcal{Y}$ is the sequence $\{P_Y^n : P_Y^n \in \mathcal{P}(\mathcal{Y}^n)\}_{n=1}^{\infty}$. Similarly, a noisy channel $\boldsymbol{W}$ with two inputs alphabets $\mathcal{X}$ and $\mathcal{Y}$ and one output alphabet $\mathcal{Z}$ is a sequence of conditional distributions $\{W^n(\cdot|\cdot,\cdot)\}_{n=1}^{\infty}$, where $W^n(\cdot|\cdot,\cdot) = \{\, W^n(\cdot|\boldsymbol{x},\boldsymbol{y}) \in \mathcal{P}(\mathcal{Z}^n)\,\}_{(\boldsymbol{x},\boldsymbol{y})\in\mathcal{X}^n\times\mathcal{Y}^n}$. Next, for $P_{X^n} \in \mathcal{P}(\mathcal{X}^n)$, $P_{Y^n} \in \mathcal{P}(\mathcal{Y}^n)$ and $\boldsymbol{z} \in \mathcal{Z}^n$, set

$$P_{X^n}P_{Y^n}W^n(\boldsymbol{z}) = \sum_{(\boldsymbol{x},\boldsymbol{y})\in\mathcal{X}^n\times\mathcal{Y}^n} P_{X^n}(\boldsymbol{x})P_{Y^n}(\boldsymbol{y})W^n(\boldsymbol{z}|\boldsymbol{x},\boldsymbol{y}), \tag{1}$$

which becomes a probability distribution on $\mathcal{Z}^n$. We denote it by $P_{X^n}P_{Y^n}W^n = \{P_{X^n}P_{Y^n}\ W^n(\boldsymbol{z})\ \}_{\boldsymbol{z}\in\mathcal{Z}^n}$. Set

$P_{Z^n} = P_{X^n} P_{Y^n} W^n$ and call P_{Z^n} *the response of* (P_{X^n}, P_{Y^n}) *through noisy channel* W^n *(or briefly the response of* (P_{X^n}, P_{Y^n})*).*

An $(n, N_1, N_2, \mu_n, \lambda_n)$ ID code for W^n is a collection $\{(P_{X^n|i}, P_{Y^n|j},\ D_{i,j}),\ i = 1, 2, \cdots, N_1,\ j = 1, 2, \cdots, N_2\}$ such that

1) $P_{X^n|i} \in \mathcal{P}(\mathcal{X}^n)\,, P_{Y^n|j} \in \mathcal{P}(\mathcal{Y}^n)\,,$
2) $D_{i,j} \subseteq \mathcal{Z}^n\,,$
3) $P_{Z^n|i,j}$ is the response of $(P_{X^n|i}, P_{Y^n|j})\,,$
4) $\mu_{n,ij} = P_{Z^n|i,j}(D^c_{i,j})\,, \mu_n = \max_{\substack{1\le i\le N_1,\\ 1\le j\le N_2}} \mu_{n,ij}\,,$
5) $\lambda_{n,ij} = \max_{(k,l)\neq(i,j)} P_{Z^n|k,l}(D_{i,j})\,, \lambda_n = \max_{\substack{1\le i\le N_1,\\ 1\le j\le N_2}} \lambda_{n,ij}\,.$

The rate of an $(n, N_1, N_2, \mu_n, \lambda_n)$ ID code is defined by

$$r_{i,n} \triangleq \frac{1}{n} \log\log N_i\,, i = 1, 2.$$

A rate pair (R_1, R_2) is said to be (μ, λ)-achievable ID rate pair if there exists an $(n, N_1,\ N_2,\ \mu_n,\ \lambda_n)$ code such that

$$\left.\begin{array}{l} \limsup\limits_{n\to\infty} \mu_n \le \mu\,, \limsup\limits_{n\to\infty} \lambda_n \le \lambda\,, \\ \liminf\limits_{n\to\infty} r_{i,n} \ge R_i\,, i = 1, 2\,. \end{array}\right\}$$

The set of all (μ, λ)-achievable ID rate pairs for $\boldsymbol{W}$ is denoted by $\mathcal{C}_{\rm ID}(\mu, \lambda|\boldsymbol{W})$, which we call the (μ, λ)-ID capacity region.

To state results for the identification capacity region, we prepare several quantities which are defined based on the notion of the *information spectrum* introduced by Han and Verdú [4].

Definition 1: For $n = 1, 2, \cdots$, let X^n and Y^n be an arbitrary prescribed independent random variable taking values in $\mathcal{X}^n$ and $\mathcal{Y}^n$, respectively. The probability mass function of X^n and Y^n is $P_{X^n}(\boldsymbol{x})$, $\boldsymbol{x} \in \mathcal{X}^n$ and $P_{Y^n}(\boldsymbol{x})$, $\boldsymbol{y} \in \mathcal{Y}^n$, respectively. A pair of two independent sources $(\boldsymbol{X}, \boldsymbol{Y})$ with alphabet $\mathcal{X}\times\mathcal{Y}$ is the sequence $\{(P_{X^n}, P_{Y^n}) : P_{X^n} \in \mathcal{P}(\mathcal{X}^n)$, $P_{Y^n} \in \mathcal{P}(\mathcal{Y}^n)\}$. A collection of such $(\boldsymbol{X}, \boldsymbol{Y})$ is denoted by $\mathcal{S}_I$. Let Z^n be an output random variable when we use X^n and Y^n as two inputs of the noisy channel W^n. In this case the joint probability mass function of (X^n, Y^n, Z^n) denoted by $P_{X^nY^nZ^n}(\boldsymbol{x}, \boldsymbol{y},\ \boldsymbol{z})$, $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \in \mathcal{X}^n \times \mathcal{Y}^n \times \mathcal{Z}^n$ is equal to $P_{X^n}(\boldsymbol{x})P_{Y^n}(\boldsymbol{y})\ W^n(\boldsymbol{z}|\boldsymbol{x}, \boldsymbol{y})$.

Definition 2: Given a joint distribution $P_{X^nY^nZ^n}(\ \boldsymbol{x},\ \boldsymbol{y},\ \boldsymbol{z}) = P_{X^n}(\boldsymbol{x})P_{Y^n}(\boldsymbol{y})\ W^n(\boldsymbol{z}|\boldsymbol{x}, \boldsymbol{y})$, the information density is the function defined on $\mathcal{X}^n \times \mathcal{Y}^n$:

$$i_{X^nY^nZ^n}(\boldsymbol{x}; \boldsymbol{z}|\boldsymbol{y}) = \log \frac{W^n(\boldsymbol{z}|\boldsymbol{x}, \boldsymbol{y})}{P_{Z^n|Y^n}(\boldsymbol{z}|\boldsymbol{y})}\,,$$
$$i_{X^nY^nZ^n}(\boldsymbol{y}; \boldsymbol{z}|\boldsymbol{x}) = \log \frac{W^n(\boldsymbol{z}|\boldsymbol{x}, \boldsymbol{y})}{P_{Z^n|X^n}(\boldsymbol{z}|\boldsymbol{x})}\,,$$
$$i_{X^nY^nZ^n}(\boldsymbol{xy}; \boldsymbol{z}) = \log \frac{W^n(\boldsymbol{z}|\boldsymbol{x}, \boldsymbol{y})}{P_{Z^n}(\boldsymbol{z})}\,.$$

Definition 3: Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of arbitrary real-valued random variables. We introduce the notion of the so-called *probabilistic limsup/inf* in the following.

$$\text{p-}\limsup_{n\to\infty} A_n \triangleq \inf\{\alpha : \lim_{n\to\infty} \Pr\{A_n \ge \alpha\} = 0\}\,,$$
$$\text{p-}\liminf_{n\to\infty} A_n \triangleq \sup\{\alpha : \lim_{n\to\infty} \Pr\{A_n \le \alpha\} = 0\}\,.$$

The *probabilistic limsup/inf* in the above definitions is considered as an extension of ordinary (deterministic) liminf. The operation of limsup/inf has the same properties as those of the operation of limsup/inf. For the details see Han and Verdú [4] and Han [9].

Definition 4: Set

$$\underline{I}(\boldsymbol{X}; \boldsymbol{Z}|\boldsymbol{Y}) \triangleq \text{p-}\liminf \frac{1}{n} i_{X^nY^nZ^n}(X^n; Z^n|Y^n),$$
$$\underline{I}(\boldsymbol{Y}; \boldsymbol{Z}|\boldsymbol{X}) \triangleq \text{p-}\liminf \frac{1}{n} i_{X^nY^nZ^n}(Y^n; Z^n|X^n),$$
$$\underline{I}(\boldsymbol{XY}; \boldsymbol{Z}) \triangleq \text{p-}\liminf \frac{1}{n} i_{X^nY^nZ^n}(X^nY^n; Z^n).$$

Furthermore, set

$$\begin{aligned} \underline{\mathcal{C}}(\boldsymbol{X}, \boldsymbol{Y}|\boldsymbol{W}) \triangleq \{(R_1, R_2) : & R_1 \le \underline{I}(\boldsymbol{X}; \boldsymbol{Z}|\boldsymbol{Y}), \\ & R_2 \le \underline{I}(\boldsymbol{Y}; \boldsymbol{Z}|\boldsymbol{X}), \\ & R_1 + R_2 \le \underline{I}(\boldsymbol{XY}; \boldsymbol{Z})\ \}\,, \end{aligned}$$
$$\underline{\mathcal{C}}(\boldsymbol{W}) \triangleq \bigcup_{(\boldsymbol{X},\boldsymbol{Y})\in\mathcal{S}_I} \underline{\mathcal{C}}(\boldsymbol{X}, \boldsymbol{Y}|\boldsymbol{W}).$$

Set

$$\overline{I}(\boldsymbol{X}; \boldsymbol{Z}|\boldsymbol{Y}) \triangleq \text{p-}\limsup \frac{1}{n} i_{X^nY^nZ^n}(X^n; Z^n|Y^n),$$
$$\overline{I}(\boldsymbol{Y}; \boldsymbol{Z}|\boldsymbol{X}) \triangleq \text{p-}\limsup \frac{1}{n} i_{X^nY^nZ^n}(Y^n; Z^n|X^n),$$
$$\overline{I}(\boldsymbol{XY}; \boldsymbol{Z}) \triangleq \text{p-}\limsup \frac{1}{n} i_{X^nY^nZ^n}(X^nY^n; Z^n).$$

Furthermore, set

$$\begin{aligned} \overline{\mathcal{C}}(\boldsymbol{X}, \boldsymbol{Y}|\boldsymbol{W}) \triangleq \{(R_1, R_2) : & R_1 \le \overline{I}(\boldsymbol{X}; \boldsymbol{Z}|\boldsymbol{Y}), \\ & R_2 \le \overline{I}(\boldsymbol{Y}; \boldsymbol{Z}|\boldsymbol{X}), \\ & R_1 + R_2 \le \overline{I}(\boldsymbol{XY}; \boldsymbol{Z})\ \}\,, \end{aligned}$$
$$\overline{\mathcal{C}}(\boldsymbol{W}) \triangleq \bigcup_{(\boldsymbol{X},\boldsymbol{Y})\in\mathcal{S}_I} \overline{\mathcal{C}}(\boldsymbol{X}, \boldsymbol{Y}|\boldsymbol{W}).$$

Han [9],[18] proved that $\underline{\mathcal{C}}(\boldsymbol{W})$ is equal to the ordinary transmission capacity region for general MACs. Han [18] proved that when $\underline{\mathcal{C}}(\boldsymbol{W}) = \overline{\mathcal{C}}(\boldsymbol{W})$, the strong converse property holds, i.e., the error probability of transmission goes to one as $n \to \infty$ for all transmission rates outside the capacity region.

Identification via multiple access channels was first investigated by Steinberg [8]. His result is the following.

Theorem A (Steinberg [8]) For general noisy channel $\boldsymbol{W}$, we have

$$\mathcal{C}_{\rm ID}(0, 0|\boldsymbol{W}) \supseteq \underline{\mathcal{C}}(\boldsymbol{W})\,. \tag{2}$$

The above theorem can be proved by an argument quite similar to the case of the identification via single-user channels. Steinberg [8] also studied the converse coding theorem. In [8] he established a new lemma useful to prove the converse coding theorem of the identification via single-user channels. Using this lemma and the capacity formula by Verdú [19], he obtained a result on the converse coding theorem for the identification via MACs.

In this paper we study the converse coding theorem for the ID via general MACs. Our approach is different from that of Steinberg [8]. We derive a function which serves as an upper bound of $1-\mu_n-\lambda_n$ for general MACs. To obtain this result we formulate a new resolvability problem for the general MAC, that is, an approximation problem of output random variables via MACs. We consider this problem and derive an upper bound of the approximation error. This upper bound is useful for analyzing the error probability of identification outside the ID capacity region.

III. Main Results

A. Definitions of Functions and their Properties

We first define several functions to describe our results and state their basic properties.

Definition 5: Let S be an arbitrary subset of $\mathcal{X}^n\times\mathcal{Y}^n\times\mathcal{Z}^n$ and $\mathbf{1}_S(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})$ be indicator functions which takes value one on S and zero outside S. Set

$$\begin{aligned}
\zeta_{n,1,S} &= \zeta_{n,1,S}(R_1,P_{X^n},P_{Y^n}|W^n)\\
&= \mathsf{E}\Big[\mathrm{e}^{-n[R_1-\frac{1}{n}i_{X^nY^nZ^n}(X^n;Z^n|Y^n)]}\\
&\qquad\qquad\times\mathbf{1}_S(X^n,Y^n,Z^n)\Big],\\
\zeta_{n,2,S} &= \zeta_{n,2,S}(R_2,P_{X^n},P_{Y^n}|W^n)\\
&= \mathsf{E}\Big[\mathrm{e}^{-n[R_2-\frac{1}{n}i_{X^nY^nZ^n}(Y^n;Z^n|X^n)]}\\
&\qquad\qquad\times\mathbf{1}_S(X^n,Y^n,Z^n)\Big],\\
\zeta_{n,3,S} &= \zeta_{n,3,S}(R_1,R_2,P_{X^n},P_{Y^n},W^n)\\
&= \mathsf{E}\Big[\Big\{\mathrm{e}^{-n[R_1-\frac{1}{n}i_{X^nZ^n}(X^n;Z^n)]}\\
&\qquad+\mathrm{e}^{-n[R_2-\frac{1}{n}i_{Y^nZ^n}(Y^n;Z^n)]}\\
&\qquad+\mathrm{e}^{-n[R_1+R_2-\frac{1}{n}i_{X^nY^nZ^n}(X^nY^n;Z^n)]}\Big\}\\
&\qquad\qquad\times\mathbf{1}_S(X^n,Y^n,Z^n)\Big].
\end{aligned}$$

Definition 6: Set

$$\begin{aligned}
T_\gamma = \{\ &(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})\in\mathcal{X}^n\times\mathcal{Y}^n\times\mathcal{Z}^n:\\
&\frac{1}{n}i_{X^nY^nZ^n}(\boldsymbol{x};\boldsymbol{z}|\boldsymbol{y})\le R_1-\gamma\,,\\
\text{or }&\frac{1}{n}i_{X^nY^nZ^n}(\boldsymbol{y};\boldsymbol{z}|\boldsymbol{x})\le R_2-\gamma\,,\\
\text{or }&\frac{1}{n}i_{X^nY^nZ^n}(\boldsymbol{x}\boldsymbol{y};\boldsymbol{z})\le R_1+R_2-2\gamma\ \}\ .
\end{aligned}$$

Define three subsets of $\mathcal{X}^n\times\mathcal{Y}^n\times\mathcal{Z}^n$ by

$$\begin{aligned}
T_{1,\gamma} &= \{\ (\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})\in\mathcal{X}^n\times\mathcal{Y}^n\times\mathcal{Z}^n:\\
&\qquad\frac{1}{n}i_{X^nY^nZ^n}(\boldsymbol{x};\boldsymbol{z}|\boldsymbol{y})\le R_1-\gamma\ \}\ ,\\
T_{2,\gamma} &= \{\ (\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})\in\mathcal{X}^n\times\mathcal{Y}^n\times\mathcal{Z}^n:\\
&\qquad\frac{1}{n}i_{X^nY^nZ^n}(\boldsymbol{y};\boldsymbol{z}|\boldsymbol{x})\le R_2-\gamma\ \}\ ,\\
T_{3,\gamma} &= \{\ (\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})\in\mathcal{X}^n\times\mathcal{Y}^n\times\mathcal{Z}^n:\\
&\qquad\frac{1}{n}i_{X^nZ^n}(\boldsymbol{x};\boldsymbol{z})\le R_1-\gamma\,,\\
&\qquad\frac{1}{n}i_{Y^nZ^n}(\boldsymbol{y};\boldsymbol{z})\le R_2-\gamma\,,\\
&\qquad\frac{1}{n}i_{X^nY^nZ^n}(\boldsymbol{x}\boldsymbol{y};\boldsymbol{z})\le R_1+R_2-2\gamma\ \}\ .
\end{aligned}$$

Set

$$\begin{aligned}
&\Omega^{(1)}_{n,i,\gamma}(R_i,P_{X^n},P_{Y^n}|W^n)\\
&=\Pr\{(X^n,Y^n,Z^n)\notin T_{i,\gamma}\}\,,i=1,2,\\
&\Omega^{(1)}_{n,3,\gamma}(R_1,R_2,P_{X^n},P_{Y^n}|W^n)\\
&=\Pr\{(X^n,Y^n,Z^n)\notin T_{3,\gamma}\}\,,\\
&\Omega^{(2)}_{n,i,\gamma}(R_i,P_{X^n},P_{Y^n}|W^n)\\
&=\zeta_{n,i,T_{i,\gamma}}(R_i,P_{X^n},P_{Y^n}|W^n),i=1,2,\\
&\Omega^{(2)}_{n,3,\gamma}(R_1,R_2,P_{X^n},P_{Y^n}|W^n)\\
&=\zeta_{n,3,T_{3,\gamma}}(R_1,R_2,P_{X^n},P_{Y^n}|W^n),\\
&\Omega_{n,i,\gamma}(R_i,P_{X^n},P_{Y^n}|W^n)\\
&=4\Omega^{(1)}_{n,i,\gamma}(R_i,P_{X^n},P_{Y^n}|W^n)\\
&\qquad+3\sqrt{\Omega^{(2)}_{n,i,\gamma}(R_i,P_{X^n},P_{Y^n}|W^n)},i=1,2,\\
&\Omega_{n,3,\gamma}(R_1,R_2,P_{X^n},P_{Y^n}|W^n)\\
&=4\Omega^{(1)}_{n,3,\gamma}(R_1,R_2,P_{X^n},P_{Y^n}|W^n)\\
&\qquad+3\sqrt{\Omega^{(2)}_{n,3,\gamma}(R_1,R_2,P_{X^n},P_{Y^n}|W^n)}.
\end{aligned}$$

Furthermore, set

$$\begin{aligned}
&\Omega_{n,\gamma}(R_1,R_2,P_{X^n},P_{Y^n}|W^n)\\
&\quad=\min\{\Omega_{n,1,\gamma}(R_1,P_{X^n},P_{Y^n}|W^n),\\
&\qquad\qquad\Omega_{n,2,\gamma}(R_2,P_{X^n},P_{Y^n}|W^n),\\
&\qquad\qquad\Omega_{n,3,\gamma}(R_1,R_2,P_{X^n},P_{Y^n}|W^n)\}
\end{aligned}$$

Finally, set

$$\begin{aligned}
&\Omega_{n,\gamma}(R_1,R_2|W^n)\\
&=\sup_{\substack{(P_{X^n},P_{Y^n})\\ \in\mathcal{P}(\mathcal{X}^n)\times\mathcal{P}(\mathcal{Y}^n)}}\Omega_{n,\gamma}(R_1,R_2,P_{X^n},P_{Y^n}|W^n)\,. \qquad (3)
\end{aligned}$$

We can prove that $\Omega_{n,\gamma}(R_1,R_2,W^n)$ and $\Omega_{n,\gamma}(R_1,R_2\ P_{X^n},P_{Y^n},W^n)$ satisfy the following two properties.

Property 1:

a) For any $0 \le \gamma < \tau$,

$$\begin{aligned}
&\Omega^{(1)}_{n,i,0}(R_i, P_{X^n}, P_{Y^n}|W^n)\\
&= \Omega^{(1)}_{n,i,\gamma}(R_i - \gamma, P_{X^n}, P_{Y^n}|W^n), i = 1, 2,\\
&\Omega^{(1)}_{n,3,0}(R_1, R_2, P_{X^n}, P_{Y^n}|W^n)\\
&= \Omega^{(1)}_{n,3,\gamma}(R_1 - \gamma, R_2 - \gamma, P_{X^n}, P_{Y^n}|W^n),\\
&\Omega^{(2)}_{n,\gamma}(R_i, P_{X^n}, P_{Y^n}|W^n)\\
&= \mathrm{e}^{-n\gamma}\Omega^{(2)}_{n,i,0}(R_i - \gamma, P_{X^n}, P_{Y^n}|W^n), i = 1, 2,\\
&\Omega^{(2)}_{n,3,\gamma}(R_1, R_2, P_{X^n}, P_{Y^n}|W^n)\\
&\le \mathrm{e}^{-n\gamma}\Omega^{(2)}_{n,3,0}(R_1 - \gamma, R_2 - \gamma, P_{X^n}, P_{Y^n}|W^n),\\
&\Omega^{(2)}_{n,i,\gamma}(R_i, P_{X^n}, P_{Y^n}|W^n) \le \mathrm{e}^{-n\gamma}, i = 1, 2,\\
&\Omega^{(2)}_{n,3,\gamma}(R_1, R_2, P_{X^n}, P_{Y^n}|W^n) \le 3\mathrm{e}^{-n\gamma},\\
&\Omega^{(2)}_{n,i,\gamma}(R_i, P_{X^n}, P_{Y^n}|W^n)\\
&\le \mathrm{e}^{-n\tau} + \Omega^{(1)}_{n,i,\tau}(R_i, P_{X^n}, P_{Y^n}|W^n)\\
&\quad -\Omega^{(1)}_{n,i,\gamma}(R_i, P_{X^n}, P_{Y^n}|W^n), i = 1, 2,\\
&\Omega^{(2)}_{n,3,\gamma}(R_1, R_2, P_{X^n}, P_{Y^n}|W^n)\\
&\le 3\mathrm{e}^{-n\tau} + \Omega^{(1)}_{n,3,\tau}(R_1, R_2, P_{X^n}, P_{Y^n}|W^n)\\
&\quad -\Omega^{(1)}_{n,3,\gamma}(R_1, R_2, P_{X^n}, P_{Y^n}|W^n).
\end{aligned}$$

b) For any $\gamma \ge 0$ and $R_1 \ge 0$, $R_2 \ge 0$,

$$\begin{aligned}
&0 \le \Omega^{(1)}_{n,i,\gamma}(R_i, P_{X^n}, P_{Y^n}|W^n) \le 1, i = 1, 2,\\
&0 \le \Omega^{(1)}_{n,3,\gamma}(R_1, R_2, P_{X^n}, P_{Y^n}|W^n) \le 1.
\end{aligned}$$

Property 2:

a) For any $\gamma \ge 0$ and $R_1, R_2 \ge 0$,

$$0 \le \Omega_{n,\gamma}(R_1, R_2, W^n) \le \frac{73}{16}.$$

b) Set

$$\begin{aligned}
&\overline{\mathcal{C}}'(\boldsymbol{X}, \boldsymbol{Y}|\boldsymbol{W})\\
&\triangleq \overline{\mathcal{C}}(\boldsymbol{X}, \boldsymbol{Y}|\boldsymbol{W})\\
&\quad \cup\{(R_1, R_2) : R_1 \le \overline{I}(\boldsymbol{X}; \boldsymbol{Z}), R_2 \le \overline{I}(\boldsymbol{Y}; \boldsymbol{Z}|\boldsymbol{X})\}\\
&\quad \cup\{(R_1, R_2) : R_1 \le \overline{I}(\boldsymbol{X}; \boldsymbol{Z}|\boldsymbol{Y}), R_2 \le \overline{I}(\boldsymbol{Y}; \boldsymbol{Z})\}\\
&\overline{\mathcal{C}}'(\boldsymbol{W}) \triangleq \bigcup_{(\boldsymbol{X},\boldsymbol{Y})\in\mathcal{S}_I} \overline{\mathcal{C}}'(\boldsymbol{X}, \boldsymbol{Y}|\boldsymbol{W}).
\end{aligned}$$

It is obvious that $\overline{\mathcal{C}}(\boldsymbol{W}) \subseteq \overline{\mathcal{C}}'(\boldsymbol{W})$. If $(R_1, R_2) \notin \overline{\mathcal{C}}'(\boldsymbol{W})$, then, there exists a small positive number γ_0 such that for any $\gamma \in [0, \gamma_0)$,

$$\lim_{n\to\infty} \Omega_{n,\gamma}(R_1, R_2|W^n) = 0.$$

Proofs of Properties 1 and 2 are quite parallel with those of Properties 1 and 2 in [13]. Proof of Property 2 part b) is given in the appendix.

B. Statement of Results

Our main result for the identification via MACs is the following.

Proposition 1: For any $(n, N_1, N_2, \mu_n, \lambda_n)$ code with $\mu_n + \lambda_n < 1$, if the rate $r_{i,n} = (1/n)\log\log N_i$ satisfies

$$r_{1,n} \ge R_1 + \frac{\log n}{n} + \frac{1}{n}\log\log(3|\mathcal{X}|)^2, \tag{4}$$

$$r_{2,n} \ge R_2 + \frac{\log n}{n} + \frac{1}{n}\log\log(3|\mathcal{Y}|)^2, \tag{5}$$

then, for any $\gamma \ge 0$, the sum $\mu_n + \lambda_n$ of two error probabilities satisfies the following:

$$1 - \mu_n - \lambda_n \le \Omega_{n,\gamma}(R_1, R_2|W^n). \tag{6}$$

From this proposition, we obtain the following corollary.

Corollary 1: For any sequence of ID codes $\{(n, N_1,, N_2, \mu_n, \lambda_n)\}_{n=1}^{\infty}$ satisfying $\mu_n + \lambda_n < 1$, $n = 1, 2, \cdots$, if

$$\liminf_{n\to\infty} r_{i,n} \ge R_i,\ i = 1, 2,$$

then, for any $\delta > 0$, there exists $n_0 = n_0(\delta)$ such that for $n \ge n_0$,

$$1 - \mu_n - \lambda_n \le \Omega_{n,\gamma}(R_1 - \delta, R_2 - \delta|W^n). \tag{7}$$

It immediately follows from Theorem A, Corollary 1 and Property 2 part b) that the following strong converse theorem holds.

Theorem 1: For any sequence of ID codes $\{(n, N_1, N_2, \mu_n, \lambda_n)\}_{n=1}^{\infty}$ satisfying $\mu_n + \lambda_n < 1$, $n = 1, 2, \cdots$, if

$$\liminf_{n\to\infty} r_{i,n} \ge R_i, i = 1, 2, \quad (R_1, R_2) \notin \overline{\mathcal{C}}'(\boldsymbol{W}),$$

then,

$$\liminf_{n\to\infty}\{\mu_n + \lambda_n\} = 1,$$

which implies that for any $\mu \ge 0, \lambda \ge 0$, $\mu + \lambda < 1$ and any noisy channel $\boldsymbol{W}$,

$$\underline{\mathcal{C}}(\boldsymbol{W}) \subseteq \mathcal{C}_{\rm ID}(\mu, \lambda|\boldsymbol{W}) \subseteq \overline{\mathcal{C}}'(\boldsymbol{W}).$$

In particular, if

$$\underline{\mathcal{C}}(\boldsymbol{W}) = \overline{\mathcal{C}}(\boldsymbol{W}) = \overline{\mathcal{C}}'(\boldsymbol{W}),$$

then, for any $\mu \ge 0, \lambda \ge 0$, $\mu + \lambda < 1$,

$$\underline{\mathcal{C}}(\boldsymbol{W}) = \mathcal{C}_{\rm ID}(\mu, \lambda|\boldsymbol{W}) = \overline{\mathcal{C}}(\boldsymbol{W}) = \overline{\mathcal{C}}'(\boldsymbol{W}).$$

Furthermore, $\mu_n + \lambda_n$ converges to one as $n \to \infty$ at rates above the ID capacity. This implies that the strong converse property holds with respect to the sum of two types of error probabilities.

C. Results for the Average Error Criterion

We have so far dealt with the case that the error probabilities of identification are measured in the maximum sense. In this subsection we consider the following average error criterion:

$$\left.\begin{aligned} \bar{\mu}_n &= \frac{1}{N_1 N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\mu_{n,ij}\,, \\ \bar{\lambda}_n &= \frac{1}{N_1 N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\lambda_{n,ij}\,. \end{aligned}\right\} \tag{8}$$

For $0 \le \mu, \lambda \le 1$, let $\mathcal{C}_{\text{ID,a}}(\mu,\lambda|\boldsymbol{W})$ be denoted by the identification capacity defined by replacing the maximum error probability criterion by the above average error probability criterion. Since $\bar{\mu}_n \le \mu_n$ and $\bar{\lambda}_n \le \lambda_n$, it is obvious that for any $\mu, \lambda \ge 0$,

$$\mathcal{C}_{\text{ID}}(\mu,\lambda|\boldsymbol{W}) \subseteq \mathcal{C}_{\text{ID,a}}(\mu,\lambda|\boldsymbol{W})\,. \tag{9}$$

We shall show that $\mathcal{C}_{\text{ID,a}}(\mu,\lambda|\boldsymbol{W})$ has the same outer bound as $\mathcal{C}_{\text{ID}}(\mu,\lambda|\boldsymbol{W})$. An important key result in the case of the average error criterion is given in the following proposition.

Proposition 2: Fix $\tau > 0$ arbitrarily. For any $(n, N_1,\ N_2,$ $\bar{\mu}_n, \bar{\lambda}_n)$ code with $\bar{\mu}_n + \bar{\lambda}_n < 1$ if the rate $r_{i,n} = \frac{1}{n}\log\log N_i$, $i = 1, 2$ satisfy

$$r_{1,n} \ge R_1 + \tau + \frac{\log n}{n} + \frac{1}{n}\log\log|\mathcal{X}|^2\,, \tag{10}$$

$$r_{2,n} \ge R_2 + \tau + \frac{\log n}{n} + \frac{1}{n}\log\log|\mathcal{Y}|^2\,, \tag{11}$$

then, for any $\gamma \ge 0$, the sum $\bar{\mu}_n + \bar{\lambda}_n$ of two average error probabilities satisfies the following:

$$\begin{aligned} 1 - \bar{\mu}_n - \bar{\lambda}_n &\le \Omega_{n,\gamma}(R_1, R_2|W^n) \\ &\quad + \nu_{n,\tau}(R_1, R_2, |\mathcal{X}|, |\mathcal{Y}|), \end{aligned}$$

where

$$\begin{aligned} &\nu_{n,\tau}(R_1, R_2, |\mathcal{X}|, |\mathcal{Y}|) \\ &\triangleq |\mathcal{X}|^{-2n(\mathrm{e}^{n\tau}-1)\mathrm{e}^{nR_1}}|\mathcal{Y}|^{-2n(\mathrm{e}^{n\tau}-1)\mathrm{e}^{nR_2}} \\ &\quad + |\mathcal{X}|^{-2n(\mathrm{e}^{n\tau}-1)\mathrm{e}^{nR_1}} \cdot |\mathcal{Y}|^{-2n(\mathrm{e}^{n\tau}-1)\mathrm{e}^{nR_2}}. \end{aligned}$$

Since $\mathrm{e}^{n\tau} - 1 \ge n\tau$, we have

$$\begin{aligned} 0 &\le \nu_{n,\tau}(R_1, R_2, |\mathcal{X}|, |\mathcal{Y}|) \\ &\le |\mathcal{X}|^{-2n^2\tau\mathrm{e}^{nR_1}} + |\mathcal{Y}|^{-2n^2\tau\mathrm{e}^{nR_2}} \\ &\quad + |\mathcal{X}|^{-2n^2\tau\mathrm{e}^{nR_1}} \cdot |\mathcal{Y}|^{-2n^2\tau\mathrm{e}^{nR_2}} \\ &\le 3|\mathcal{X}|^{-2n^2\tau\mathrm{e}^{nR_1}} \cdot |\mathcal{Y}|^{-2n^2\tau\mathrm{e}^{nR_2}}. \end{aligned}$$

which implies that for each fixed $\tau > 0$, $\nu_{n,\tau}(R_1,\ R_2,\ |\mathcal{X}|,$ $|\mathcal{Y}|)$ decays double exponentially as n tends to infinity.

From this proposition, we obtain the following corollary.

Corollary 2: For any sequence of ID codes $\{(n,\ N_1,,\ N_2,$ $\bar{\mu}_n,\ \bar{\lambda}_n)\ \}_{n=1}^{\infty}$ satisfying $\bar{\mu}_n + \bar{\lambda}_n < 1$, $n = 1, 2, \cdots$, if

$$\liminf_{n\to\infty} r_{i,n} \ge R_i,\ i = 1, 2,$$

then, for any $\delta > 0$, there exists $n_0 = n_0(\delta)$ such that for $n \ge n_0$,

$$\begin{aligned} 1 - \bar{\mu}_n - \bar{\lambda}_n &\le \Omega_{n,\gamma}(R_1 - \delta, R_2 - \delta|W^n) \\ &\quad + \nu_{n,\tau}(R_1 - \delta, R_2 - \delta, |\mathcal{X}|, |\mathcal{Y}|)\,. \end{aligned} \tag{12}$$

It immediately follows from Theorem A, Corollary 2 and Property 2 part b) that the following strong converse theorem holds.

Theorem 2: For any sequence of ID codes $\{(n, N_1,\ N_2,\ \bar{\mu}_n,$ $\bar{\lambda}_n)\ \}_{n=1}^{\infty}$ satisfying $\bar{\mu}_n + \bar{\lambda}_n < 1$, $n = 1, 2, \cdots$, if

$$\liminf_{n\to\infty} r_{i,n} \ge R_i, i = 1, 2, \quad (R_1, R_2) \notin \overline{\mathcal{C}}'(\boldsymbol{W})\,,$$

then,

$$\liminf_{n\to\infty}\{\bar{\mu}_n + \bar{\lambda}_n\} = 1,$$

which implies that for any $\mu \ge 0, \lambda \ge 0,\ \mu + \lambda < 1$ and any noisy channel $\boldsymbol{W}$,

$$\underline{\mathcal{C}}(\boldsymbol{W}) \subseteq \mathcal{C}_{\text{ID}}(\mu,\lambda|\boldsymbol{W}) \subseteq \mathcal{C}_{\text{ID,a}}(\mu,\lambda|\boldsymbol{W}) \subseteq \overline{\mathcal{C}}'(\boldsymbol{W})\,.$$

In particular, if

$$\underline{\mathcal{C}}(\boldsymbol{W}) = \overline{\mathcal{C}}(\boldsymbol{W}) = \overline{\mathcal{C}}'(\boldsymbol{W}),$$

then, for any $\mu \ge 0, \lambda \ge 0,\ \mu + \lambda < 1$,

$$\underline{\mathcal{C}}(\boldsymbol{W}) = \mathcal{C}_{\text{ID}}(\mu,\lambda|\boldsymbol{W}) = \mathcal{C}_{\text{ID,a}}(\mu,\lambda|\boldsymbol{W}) = \overline{\mathcal{C}}(\boldsymbol{W}) = \overline{\mathcal{C}}'(\boldsymbol{W}).$$

Furthermore, $\bar{\mu}_n + \bar{\lambda}_n$ converges to one as $n \to \infty$ at rates above the ID capacity. This implies that the strong converse property holds with respect to the sum of two types of error probabilities.

IV. Proof of Results

In this section we shall give the proofs of the results stated in the previous section.

For the proofs of Propositions 1 and 2, we first formulate a new resolvability problem for the general MAC, that is, an approximation problem of output random variables via MACs. We consider this problem and derive an upper bound of the approximation error. This upper bound is useful for analyzing the error probability of identification outside the ID capacity region. Next, we prove Propositions 1 and 2 based on a new method of converting the direct coding theorem for the MAC resolvability problem into the converse coding theorem of the ID via MACs. Han and Verdú [4] provided a method of converting the direct coding theorem for the channel resolvability problem into the converse coding theorem of the ID channel. Our method is an extension of their method in the case of MACs.

A. MAC Resolvability Problem

Definition 7: Let $U_{M_i}, i = 1, 2$ be the uniform random variables taking values in $\mathcal{U}_{M_1} = \{1, 2,\ \cdots,\ M_i\}$. By two maps $\tilde{\varphi}_1 : \mathcal{U}_{M_1} \to \mathcal{X}^n$ and $\tilde{\varphi}_2 : \mathcal{U}_{M_2} \to \mathcal{Y}^n$, the uniform random variables U_{M_1} and U_{M_2} is transformed into the random variable $\tilde{X}^n = \tilde{\varphi}_1(U_{M_1})$ and $\tilde{Y}^n = \tilde{\varphi}_2(U_{M_2})$, respectively. Let $\mathcal{P}_{M_1}\ (\mathcal{X}^n)$ and $\mathcal{P}_{M_2}\ (\mathcal{Y}^n)$ be sets of all probability

distributions on $\mathcal{X}^n$ that can be created by the transformation of U_{M_1} and U_{M_2}. Elements of $\mathcal{P}_{M_1}(\mathcal{X}^n)$ and $\mathcal{P}_{M_2}(\mathcal{Y}^n)$, respectively are called M_1 and M_2-types. Every random variable $\tilde{X}^n = \tilde{\varphi}_1(U_{M_1})$ created by some transformation map $\tilde{\varphi}_1 : \mathcal{U}_{M_n} \to \mathcal{X}^n$ and U_{M_1} has M_1-type. Similarly, every random variable $\tilde{Y}^n = \tilde{\varphi}_2(U_{M_2})$ created by some transformation map $\tilde{\varphi}_2 : \mathcal{U}_{M_2} \to \mathcal{Y}^n$ and U_{M_2} has M_2-type.

Definition 8: For $\tilde{\varphi}_1 : \mathcal{U}_{M_1} \to \mathcal{X}^n$ and $\tilde{\varphi}_2 : \mathcal{U}_{M_2} \to \mathcal{Y}^n$, define $P_{\tilde{X}^n} = P_{\tilde{\varphi}_1(U_{M_1})}$ and $P_{\tilde{Y}^n} = P_{\tilde{\varphi}_2(U_{M_2})}$. We use $P_{\tilde{X}^n}$ and $P_{\tilde{Y}^n}$ as approximations of X^n and Y^n, respectively. Let $\tilde{Q}^{(1)}$ be a response of ($P_{\tilde{X}^n}$,P_{Y^n}) and let $\tilde{Q}^{(2)}$ be a response of (P_{X^n},$P_{\tilde{Y}^n}$). Let $\tilde{Q}^{(3)}$ be a response of ($P_{\tilde{X}^n}$,$P_{\tilde{Y}^n}$). Set

$$\underline{\tilde{Q}} \stackrel{\triangle}{=} (\tilde{Q}^{(1)}, \tilde{Q}^{(2)}, \tilde{Q}^{(3)}).$$

Let $\tilde{\mathcal{Q}}^{(t)}$, $t = 1, 2, 3$, be sets of all responses $\tilde{Q}^{(t)}$.

The following is a lemma on the cardinalities of $\mathcal{P}_{M_1}(\mathcal{X}^n)$, $\mathcal{P}_{M_2}(\mathcal{Y}^n)$ and $\tilde{\mathcal{Q}}^{(t)}$, $t = 1, 2, 3$.

Lemma 1:

a)

$$|\mathcal{P}_{M_1}(\mathcal{X}^n)| \le |\mathcal{X}|^{nM_1}, |\mathcal{P}_{M_2}(\mathcal{Y}^n)| \le |\mathcal{Y}|^{nM_2}.$$

b)

$$|\tilde{\mathcal{Q}}^{(1)}| \le |\mathcal{P}_{M_1}(\mathcal{X}^n)|, |\tilde{\mathcal{Q}}^{(2)}| \le |\mathcal{P}_{M_2}(\mathcal{Y}^n)| ,$$
$$|\tilde{\mathcal{Q}}^{(3)}| \le |\mathcal{P}_{M_1}(\mathcal{X}^n)||\mathcal{P}_{M_2}(\mathcal{Y}^n)|.$$

Now we use $\underline{\tilde{Q}}$ as an approximation of Q. In this case we are interested in the asymptotic behavior of the following triple of approximation errors

$$(d(Q, \tilde{Q}^{(1)}), d(Q, \tilde{Q}^{(2)}), d(Q, \tilde{Q}^{(3)}))$$

measured by the variational distance. We shall derive explicit upper bounds of $d(Q, \tilde{Q}^{(t)}), t = 1, 2, 3$. This result is a mathematical core of the converse coding theorem for the ID via MACs.

Lemma 2: Set $M_t = \lceil e^{nR_t} \rceil$, $t = 1, 2$, where $\lceil a \rceil$ is the minimum integer not below a. Let $S_i, i = 1, 2, 3$ be arbitrary prescribed subsets of $\mathcal{X}^n \times \mathcal{Y}^n \times \mathcal{Z}^n$. Let (X^n, Y^n) be a pair of two independent random variables with distribution (P_{X^n}, P_{Y^n}). Let Q be a response of (P_{X^n}, P_{Y^n}). Then, for any (P_{X^n}, P_{Y^n}) and its response Q, there exist $\tilde{\varphi}_1 : \mathcal{U}_{M_1} \to \mathcal{X}^n$ and $\tilde{\varphi}_2 : \mathcal{U}_{M_2} \to \mathcal{Y}^n$ such that the three variational distances $d(Q, \tilde{Q}^{(t)})$, $t = 1, 2, 3$ satisfies the following:

$$\begin{aligned}&d(Q, \tilde{Q}^{(t)})\\ &\le 4\mathsf{E}\left[\mathbf{1}_{S_t^c}(X^n, Y^n, Z^n)\right] + 3\sqrt{\zeta_{n,t,S_t}}, \text{ for } t = 1, 2, 3.\end{aligned}$$

The proof of the above lemma is given in the appendix.

B. Proofs of Propositions and Corollaries

In this subsection we prove Propositions 1 and 2 and Corollaries 1 and 2 stated in the previous section. We first prove Propositions 1 and 2 using Lemmas 1 and 2. Next we prove Corollaries 1 and 2 respectively, using Propositions 1 and 2.

Proof of Proposition 1: Let $P_{X^n|i} \in \mathcal{P}(\mathcal{X}^n), i \in \mathcal{N}_1$ and $P_{Y^n|j} \in \mathcal{P}(\mathcal{Y}^n), j \in \mathcal{N}_2$, be codewords of $(n, N_1, N_2, \mu_n, \lambda_n)$ code of the ID channel and $D_{i,j} \subseteq \mathcal{Z}^n$, $i \in \mathcal{N}_1$, $j \in \mathcal{N}_2$ be decoding regions corresponding to the codewords. Let the response $P_{X^n|i}P_{Y^n|i}W^n$ of ($P_{X^n|i}$, $P_{Y^n|j}$) be denoted by Q_{ij}. We choose $S_i = T_{i,\gamma}$, $i = 1, 2, 3$. Then, by Lemma 2, there exists $\underline{\tilde{Q}}$ such that

$$d(Q_{ij}, \tilde{Q}^{(t)}) \le \eta_{n,t}(P_{X^n}, P_{Y^n}), t = 1, 2, 3, \tag{13}$$

where we put

$$\eta_{n,t}(P_{X^n}, P_{Y^n}) \stackrel{\triangle}{=} \Omega_{n,t,\gamma}(R_t, P_{X^n}, P_{Y^n}|W^n), t = 1, 2,$$
$$\eta_{n,3}(P_{X^n}, P_{Y^n}) \stackrel{\triangle}{=} \Omega_{n,3,\gamma}(R_1, R_2, P_{X^n}, P_{Y^n}|W^n).$$

For simplicity of notation we set $\eta_n \stackrel{\triangle}{=} \Omega_{n,\gamma}(R_1, R_2|W^n)$. Then by the definition of $\Omega_{n,\gamma}(R_1, R_2|W^n)$, we have

$$\eta_n = \sup_{\substack{(P_{X^n}, P_{Y^n}) \\ \in \mathcal{P}(\mathcal{X}^n) \times \mathcal{P}(\mathcal{Y}^n)}} \min_{t=1,2,3} \{\eta_{n,t}(P_{X^n}, P_{Y^n})\}. \tag{14}$$

From (13) and (14), it follows that for any Q_{ij}, there exists $t \in \{1, 2, 3\}$ and $\tilde{Q}^{(t)} \in \tilde{\mathcal{Q}}^{(t)}$ such that $d(Q_{ij}, \tilde{Q}^{(t)}) \le \eta_n$. Define

$$\begin{aligned}\mathcal{L}^{(t)} \stackrel{\triangle}{=} \{&(i, j) \in \mathcal{N}_1 \times \mathcal{N}_2 :\\ &d(Q_{ij}, \tilde{Q}^{(t)}) \le \eta_n \text{ for some } \tilde{Q}^{(t)} \in \tilde{\mathcal{Q}}^{(t)}\}.\end{aligned}$$

Since

$$\mathcal{L}^{(1)} \cup \mathcal{L}^{(2)} \cup \mathcal{L}^{(3)} = \mathcal{N}_1 \times \mathcal{N}_2 \,,$$

we have

$$|\mathcal{L}^{(t)}| \ge \frac{1}{3} N_1 N_2 \text{ for some } t \in \{1, 2, 3\}. \tag{15}$$

Set $a_t \stackrel{\triangle}{=} |\tilde{\mathcal{Q}}^{(t)}|$, $t = 1, 2, 3$. Note that $M_t \le 2e^{nR_t}$, $t = 1, 2$. Then by Lemma 1, we have

$$a_1 \le |\mathcal{X}|^{2e^{nR_1}}, a_2 \le |\mathcal{Y}|^{2e^{nR_2}}, a_3 \le |\mathcal{X}|^{2e^{nR_1}} \cdot |\mathcal{Y}|^{2e^{nR_2}}.$$

Set

$$b_1 \stackrel{\triangle}{=} |\mathcal{X}|^{2e^{nR_1}}, b_2 \stackrel{\triangle}{=} |\mathcal{Y}|^{2e^{nR_2}}, b_3 \stackrel{\triangle}{=} |\mathcal{X}|^{2e^{nR_1}} \cdot |\mathcal{Y}|^{2e^{nR_2}}.$$

Now, we suppose that the inequality (15) holds for $t = 1$. Set

$$\mathcal{L}^{(1)}_{1|2}(j) \stackrel{\triangle}{=} \{i : (i, j) \in \mathcal{L}^{(1)}\}.$$

Then, we have

$$|\mathcal{L}^{(1)}_{1|2}(j)| \ge \frac{1}{3} N_1 \text{ for some } j.$$

Then if

$$\frac{1}{3} N_1 \ge 3^{2e^{nR_1} - 1} \cdot b_1 \ge 3b_1 > a_1 = |\tilde{\mathcal{Q}}^{(1)}|$$

or equivalent to

$$r_{1,n} \ge R_1 + \frac{\log n}{n} + \frac{1}{n} \log\log(3|\mathcal{X}|)^2,$$

there exist two pairs (i, j) and (k, j), $i \ne k$ and $\tilde{Q}^{(1)} \in \tilde{\mathcal{Q}}^{(1)}$ such that

$$d(Q_{ij}, \tilde{Q}^{(1)}) \le \eta_n \, , d(Q_{kj}, \tilde{Q}^{(1)}) \le \eta_n \, .$$

For the above two pairs, we have

$$d(Q_{ij},Q_{kj}) \le d(Q_{ij},\tilde{Q}^{(1)}) + d(Q_{kj},\tilde{Q}^{(1)}) \le 2\eta_n. \quad (16)$$

On the other hand, we have

$$\begin{aligned} d(Q_{ij},Q_{kj}) &\ge 2\left[Q_{ij}(D_{i,j}) - Q_{kj}(D_{i,j})\right] \\ &\ge 2\left(1-\mu_n-\lambda_n\right), \end{aligned}$$

which together with (16) yields that $1-\mu_n-\lambda_n \le \eta_n$. Next, we suppose that the inequality (15) holds for $t=2$. By an argument quite similar to the previous one, we can prove that if

$$\frac{1}{3}N_2 \ge 3^{2\mathrm{e}^{nR_2}-1}\cdot b_2 \ge 3b_2 > a_2 = |\tilde{\mathcal{Q}}^{(2)}|$$

or equivalent to

$$r_{2,n} \ge R_2 + \frac{\log n}{n} + \frac{1}{n}\log\log(3|\mathcal{Y}|)^2,$$

we have $1-\mu_n-\lambda_n \le \eta_n$. Finally, we suppose that the inequality (15) holds for $t=3$. Since

$$\frac{1}{3}N_1N_2 \ge 3^{2(\mathrm{e}^{nR_1}+\mathrm{e}^{nR_2})-1}\cdot b_1b_2 \ge 3b_1b_2 > a_1a_2 \ge |\tilde{\mathcal{Q}}^{(3)}|$$

there exist two pairs (i,j) and (k,l), $(i,j)\ne(k,l)$ and $\tilde{Q}^{(3)} \in \tilde{\mathcal{Q}}^{(3)}$ such that

$$d(Q_{ij},\tilde{Q}^{(3)}) \le \eta_n, d(Q_{kl},\tilde{Q}^{(3)}) \le \eta_n.$$

For the above two pairs, we have

$$d(Q_{ij},Q_{kl}) \le d(Q_{ij},\tilde{Q}^{(3)}) + d(Q_{kl},\tilde{Q}^{(3)}) \le 2\eta_n. \quad (17)$$

On the other hand, we have

$$\begin{aligned} d(Q_{ij},Q_{kl}) &\ge 2\left[Q_{ij}(D_{i,j}) - Q_{kl}(D_{i,j})\right] \\ &\ge 2\left(1-\mu_n-\lambda_n\right), \end{aligned}$$

which together with (17) yields that $1-\mu_n-\lambda_n \le \eta_n$. This completes the proof of Proposition 1. ■

Proof of Proposition 2: Let $P_{X^n|i} \in \mathcal{P}(\mathcal{X}^n), i \in \mathcal{N}_1$, and $P_{Y^n|j} \in \mathcal{P}(\mathcal{Y}^n), j \in \mathcal{N}_2$, be codewords of $(n,N_1,N_2,\bar{\mu}_n,\bar{\lambda}_n)$ code of the ID channel and $D_{ij} \subseteq \mathcal{Z}^n$, $i \in \mathcal{N}_1$, $j \in \mathcal{N}_2$ be decoding regions corresponding to the codewords. Let the response $P_{X^n|i}P_{Y^n|i}W^n$ of $(P_{X^n|i},\ P_{Y^n|j})$ be denoted by Q_{ij}. For $\tilde{Q}^{(t)} \in \tilde{\mathcal{Q}}^{(t)}$, $t=1,2,3$, define

$$\mathcal{S}_t(\tilde{Q}^{(t)}) \triangleq \left\{(i,j) \in \mathcal{N}_1\times\mathcal{N}_2 : d(Q_{ij},\tilde{Q}^{(t)}) \le \eta_n\right\}.$$

For $t=1,2,3$, set

$$\tilde{\mathcal{Q}}_0^{(t)} \triangleq \left\{\tilde{Q}^{(t)} \in \tilde{\mathcal{Q}}^{(t)} : |\mathcal{S}_t(\tilde{Q}^{(t)})| \ge 1\right\}.$$

Then, the validity of Lemma 2 implies that

$$\mathcal{L}^{(t)} = \bigcup_{\tilde{Q}^{(t)}\in\tilde{\mathcal{Q}}_0^{(t)}} \mathcal{S}_t(\tilde{Q}^{(t)}) \text{ for } t=1,2,3,$$

$$\bigcup_{t=1}^{3}\bigcup_{\tilde{Q}^{(t)}\in\tilde{\mathcal{Q}}_0^{(t)}} \mathcal{S}_t(\tilde{Q}^{(t)}) = \mathcal{N}_1\times\mathcal{N}_2.$$

Define

$$\tilde{\mathcal{Q}}_1^{(1)} \triangleq \Big\{\tilde{Q}^{(1)} \in \tilde{\mathcal{Q}}^{(1)} : \mathcal{S}_1(\tilde{Q}^{(1)}) \text{ consists of pairs } (i,j) \text{ such that for fixed } j \text{ we have only one index } i\Big\},$$

$$\tilde{\mathcal{Q}}_2^{(1)} \triangleq \Big\{\tilde{Q}^{(1)} \in \tilde{\mathcal{Q}}^{(1)} : \mathcal{S}_1(\tilde{Q}^{(1)}) \text{ consists of pairs } (i,j) \text{ such that for fixed } j \text{ we have more than two indexes } i\Big\},$$

$$\tilde{\mathcal{Q}}_1^{(2)} \triangleq \Big\{\tilde{Q}^{(2)} \in \tilde{\mathcal{Q}}^{(2)} : \mathcal{S}_2(\tilde{Q}^{(2)}) \text{ consists of pairs } (i,j) \text{ such that for fixed } i \text{ we have only one index } j\Big\},$$

$$\tilde{\mathcal{Q}}_2^{(2)} \triangleq \Big\{\tilde{Q}^{(2)} \in \tilde{\mathcal{Q}}^{(2)} : \mathcal{S}_2(\tilde{Q}^{(2)}) \text{ consists of pairs } (i,j) \text{ such that for fixed } i \text{ we have more than two indexes } j\Big\},$$

$$\tilde{\mathcal{Q}}_1^{(3)} \triangleq \Big\{\tilde{Q}^{(3)} \in \tilde{\mathcal{Q}}^{(3)} : \mathcal{S}_3(\tilde{Q}^{(3)}) \text{ consists of pairs } (i,j) \text{ with one index pair } (i,j)\Big\},$$

$$\tilde{\mathcal{Q}}_2^{(3)} \triangleq \Big\{\tilde{Q}^{(3)} \in \tilde{\mathcal{Q}}^{(3)} : \mathcal{S}_3(\tilde{Q}^{(3)}) \text{ consists of more than two index pairs } (i,j)\Big\}.$$

It is obvious that

$$\tilde{\mathcal{Q}}_1^{(t)} \cup \tilde{\mathcal{Q}}_2^{(t)} = \tilde{\mathcal{Q}}_0^{(t)},\ t=1,2,3.$$

Observe that if $\tilde{Q}^{(1)} \in \tilde{\mathcal{Q}}_2^{(1)}$, for any $(i,j) \in \mathcal{S}_1(\tilde{Q})$, there exists an index $k \ne i$ such that $(k,j) \in \mathcal{S}_1(\tilde{Q})$. Then, we have

$$\begin{aligned} &1-\mu_{n,ij}-\lambda_{n,ij} \\ &\le \left[Q_{ij}(D_{i,j}) - Q_{kj}(D_{k,j})\right] \le (1/2)d(Q_{ij},Q_{kj}) \\ &\le (1/2)\left[d(Q_{ij},\tilde{Q}^{(1)}) + d(Q_{kj},\tilde{Q}^{(1)})\right] \le \eta_n. \end{aligned} \quad (18)$$

Similarly, if $\tilde{Q}^{(2)} \in \tilde{\mathcal{Q}}_2^{(2)}$, for any $(i,j) \in \mathcal{S}_2(\tilde{Q}^{(2)})$, there exists an index $l \ne j$ such that $(i,l) \in \mathcal{S}_2(\tilde{Q}^{(2)})$. Then, we have

$$1-\mu_{n,ij}-\lambda_{n,ij} \le \eta_n. \quad (19)$$

If $\tilde{Q}^{(3)} \in \tilde{\mathcal{Q}}_2^{(3)}$, for any $(i,j) \in \mathcal{S}_3(\tilde{Q}^{(3)})$, there exists an index $(i,j) \ne (k,l)$ such that $(k,l) \in \mathcal{S}_3(\tilde{Q}^{(3)})$. Then, we have

$$1-\mu_{n,ij}-\lambda_{n,ij} \le \eta_n. \quad (20)$$

We obtain the following chain of inequalities:

$$\begin{aligned} &1-\bar{\mu}_n-\bar{\lambda}_n \\ &= \frac{1}{N_1N_2}\sum_{(i,j)\in\mathcal{N}_1\times\mathcal{N}_2}(1-\mu_{n,ij}-\lambda_{n,ij}) \\ &\le \frac{1}{N_1N_2}\sum_{t=1}^{3}\sum_{(i,j)\in\mathcal{L}^{(t)}}(1-\mu_{n,ij}-\lambda_{n,ij}) \\ &= \frac{1}{N_1N_2}\sum_{t=1}^{3}\sum_{\tilde{Q}^{(t)}\in\tilde{\mathcal{Q}}_0^{(t)}}\sum_{(i,j)\in\mathcal{S}_t(\tilde{Q}^{(t)})}(1-\mu_{n,ij}-\lambda_{n,ij}) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{N_1N_2}\sum_{t=1}^{3}\sum_{\tilde{Q}^{(t)}\in\tilde{\mathcal{Q}}_1^{(t)}}\sum_{(i,j)\in\mathcal{S}_t(\tilde{Q}^{(t)})}(1-\mu_{n,ij}-\lambda_{n,ij})\\
&\quad+\frac{1}{N_1N_2}\sum_{t=1}^{3}\sum_{\tilde{Q}^{(t)}\in\tilde{\mathcal{Q}}_2^{(t)}}\sum_{(i,j)\in\mathcal{S}_t(\tilde{Q}^{(t)})}(1-\mu_{n,ij}-\lambda_{n,ij})\\
&\overset{\text{(a)}}{\le} \sum_{t=1}^{2}\frac{|\tilde{\mathcal{Q}}_1^{(t)}|}{N_t}+\frac{|\tilde{\mathcal{Q}}_1^{(3)}|}{N_1N_2}+\eta_n\\
&\le \frac{|\mathcal{P}_{M_1}(\mathcal{X}^n)|}{N_1}+\frac{|\mathcal{P}_{M_2}(\mathcal{Y}^n)|}{N_2}\\
&\qquad+\frac{|\mathcal{P}_{M_1}(\mathcal{X}^n)||\mathcal{P}_{M_2}(\mathcal{Y}^n)|}{N_1N_2}+\eta_n\\
&\overset{\text{(b)}}{\le} \frac{|\mathcal{X}|^{2n\mathrm{e}^{nR_1}}}{N_1}+\frac{|\mathcal{Y}|^{2n\mathrm{e}^{nR_2}}}{N_2}\\
&\qquad+\frac{|\mathcal{X}|^{2n\mathrm{e}^{nR_1}}|\mathcal{Y}|^{2n\mathrm{e}^{nR_2}}}{N_1N_2}+\eta_n. \qquad (21)
\end{aligned}
$$

Step (a) follows from (18) -(20). Step (b) follows from Lemma 1 and $M_t \le 2\mathrm{e}^{nR_t}, t=1,2$. Then, if $N_1 \ge |\mathcal{X}|^{2n\mathrm{e}^{n(R_1+\tau)}}$ and $N_2 \ge |\mathcal{Y}|^{2n\mathrm{e}^{n(R_2+\tau)}}$ or equivalent to

$$
\begin{aligned}
r_{1,n} &\ge R_1+\tau+\frac{\log n}{n}+\frac{1}{n}\log\log|\mathcal{X}|^2,\\
r_{2,n} &\ge R_2+\tau+\frac{\log n}{n}+\frac{1}{n}\log\log|\mathcal{Y}|^2,
\end{aligned}
$$

from (21), we have

$$
\begin{aligned}
&1-\bar{\mu}_n-\bar{\lambda}_n\\
&\le \frac{|\mathcal{X}|^{2n\mathrm{e}^{nR_1}}}{|\mathcal{X}|^{2n\mathrm{e}^{n(R_1+\tau)}}}+\frac{|\mathcal{Y}|^{2n\mathrm{e}^{nR_2}}}{|\mathcal{Y}|^{2n\mathrm{e}^{n(R_2+\tau)}}}\\
&\quad+\frac{|\mathcal{X}|^{2n\mathrm{e}^{nR_1}}|\mathcal{Y}|^{2n\mathrm{e}^{nR_2}}}{|\mathcal{X}|^{2n\mathrm{e}^{n(R_1+\tau)}}|\mathcal{Y}|^{2n\mathrm{e}^{n(R_2+\tau)}}}+\eta_n\\
&=\nu_{n,\tau}(R_1,R_2,|\mathcal{X}|,|\mathcal{Y}|)+\Omega_{n,\gamma}(R_1,R_2,W^n).
\end{aligned}
$$

This completes the proof of Proposition 2. ■

Proof of Corollary 1: We assume that a sequence of ID codes $\{(n,N_1,N_2,\ \mu_n,\ \lambda_n)\ \}_{n=1}^{\infty}$ satisfies $\mu_n+\lambda_n<1, n=1,2,\cdots$, and

$$
\liminf_{n\to\infty}\frac{1}{n}\log\log N_i \ge R_i, i=1,2. \qquad (22)
$$

Since

$$
\begin{aligned}
\lim_{n\to\infty}\left[\frac{\log n}{n}+\frac{1}{n}\log\log(3|\mathcal{X}|)^2\right]&=0,\\
\lim_{n\to\infty}\left[\frac{\log n}{n}+\frac{1}{n}\log\log(3|\mathcal{Y}|)^2\right]&=0,
\end{aligned}
$$

there exists $n_1=n_1(\delta,|\mathcal{X}|,|\mathcal{Y}|)$ such that for any $n\ge n_1$

$$
\begin{aligned}
\frac{\log n}{n}+\frac{1}{n}\log\log(3|\mathcal{X}|)^2 &\le \frac{\delta}{2},\\
\frac{\log n}{n}+\frac{1}{n}\log\log(3|\mathcal{Y}|)^2 &\le \frac{\delta}{2}.
\end{aligned}
$$

On the other hand, by virtue of (22), there exists $n_2=n_2(\delta)$ such that for any $n\ge n_2$

$$
\frac{1}{n}\log\log N_i \ge R_i-\frac{\delta}{2}, i=1,2.
$$

Set $n_0=n_0(\delta,|\mathcal{X}|)=\max\{n_1,n_2\}$. Then, for any $n\ge n_0$, we have

$$
\begin{aligned}
\frac{1}{n}\log\log N_1 &\ge R_1-\delta+\frac{\log n}{n}+\frac{1}{n}\log\log(3|\mathcal{X}|)^2,\\
\frac{1}{n}\log\log N_2 &\ge R_2-\delta+\frac{\log n}{n}+\frac{1}{n}\log\log(3|\mathcal{Y}|)^2.
\end{aligned}
$$

Applying Proposition 1 with respect to $R_i-\delta, i=1,2$, for $n\ge n_0$, we have (7) of Corollary 1. ■

Proof of Corollary 2: We assume that a sequence of ID codes $\{(n,N_1,N_2,\ \bar{\mu}_n,\ \bar{\lambda}_n)\ \}_{n=1}^{\infty}$ satisfies $\bar{\mu}_n+\bar{\lambda}_n<1, n=1,2,\cdots$, and

$$
\liminf_{n\to\infty}\frac{1}{n}\log\log N_i \ge R_i, i=1,2. \qquad (23)
$$

We choose $\tau=(1/3)\delta$. Since

$$
\begin{aligned}
\lim_{n\to\infty}\left[\frac{\log n}{n}+\frac{1}{n}\log\log|\mathcal{X}|^2\right]&=0,\\
\lim_{n\to\infty}\left[\frac{\log n}{n}+\frac{1}{n}\log\log|\mathcal{Y}|^2\right]&=0,
\end{aligned}
$$

there exists $n_1=n_1(\delta,|\mathcal{X}|,|\mathcal{Y}|)$ such that for any $n\ge n_1$

$$
\begin{aligned}
\tau+\frac{\log n}{n}+\frac{1}{n}\log\log|\mathcal{X}|^2 &\le \frac{\delta}{2},\\
\tau+\frac{\log n}{n}+\frac{1}{n}\log\log|\mathcal{Y}|^2 &\le \frac{\delta}{2}.
\end{aligned}
$$

On the other hand, by virtue of (23), there exists $n_2=n_2(\delta)$ such that for any $n\ge n_2$

$$
\frac{1}{n}\log\log N_i \ge R_i-\frac{\delta}{2}, i=1,2.
$$

Set $n_0=n_0(\delta,|\mathcal{X}|)=\max\{n_1,n_2\}$. Then, for any $n\ge n_0$, we have

$$
\begin{aligned}
\frac{1}{n}\log\log N_1 &\ge R_1-\delta+\tau+\frac{\log n}{n}+\frac{1}{n}\log\log|\mathcal{X}|^2,\\
\frac{1}{n}\log\log N_2 &\ge R_2-\delta+\tau+\frac{\log n}{n}+\frac{1}{n}\log\log|\mathcal{Y}|^2.
\end{aligned}
$$

Applying Proposition 1 with respect to $R_i-\delta, i=1,2$, for $n\ge n_0$, we have (12) of Corollary 2. ■

APPENDIX

A. *Proof of Property 2*

Proof of Property 2 part b): We assume that $(R_1,R_2)\notin\overline{\mathcal{C}}'(\boldsymbol{W})$. Then there exists small positive number γ_0 such that for any $0\le\gamma\le\gamma_0$, we have

$$
(R_1-\gamma,R_2-\gamma)\notin\overline{\mathcal{C}}'(\boldsymbol{W}).
$$

Then, by the definition of $\overline{\mathcal{C}}'(\boldsymbol{W})$, for any $(\boldsymbol{X},\,\boldsymbol{Y})\in\mathcal{S}_I$, we have

$$(R_1-\gamma,R_2-\gamma)\notin\overline{\mathcal{C}}'(\boldsymbol{X},\boldsymbol{Y}|\boldsymbol{W})\,,$$

or equivalent to

$$R_1-\gamma>\overline{I}(\boldsymbol{X};\boldsymbol{Z}|\boldsymbol{Y}), \tag{24}$$

$$\text{or}\quad R_2-\gamma>\overline{I}(\boldsymbol{Y};\boldsymbol{Z}|\boldsymbol{X}), \tag{25}$$

$$\text{or}\quad \begin{cases} R_1-\gamma>\overline{I}(\boldsymbol{X};\boldsymbol{Z}), R_2-\gamma>\overline{I}(\boldsymbol{Y};\boldsymbol{Z}),\\ R_1+R_2-2\gamma>\overline{I}(\boldsymbol{XY};\boldsymbol{Z}).\end{cases} \tag{26}$$

We first assume that (24) holds. Then by the definition of $\overline{I}(\boldsymbol{X};\boldsymbol{Z}|\boldsymbol{Y})$, for any $\gamma\in[0,\gamma_0)$,

$$\liminf_{n\to\infty}\Omega^{(1)}_{n,1,\gamma}(R_1,P_{X^n},P_{Y^n}|W^n)=0. \tag{27}$$

We choose τ so that $\tau=(1/2)(\,\gamma+\gamma_0)$. Then by Property 1 part a), we have

$$\begin{aligned}&\Omega^{(2)}_{n,t,\gamma}(R_1,P_{X^n},P_{Y^n}|W^n)\\ &\quad\le \mathrm{e}^{-n\tau}+\Omega^{(1)}_{n,t,\tau}(R_t,P_{X^n},P_{Y^n}|W^n)\\ &\qquad-\Omega^{(1)}_{n,t,\gamma}(R_1,P_{X^n},P_{Y^n}|W^n).\end{aligned} \tag{28}$$

From (27) and (28), for any $\gamma\in[0,\gamma_0)$,

$$\liminf_{n\to\infty}\Omega_{n,1,\gamma}(R_1,P_{X^n},P_{Y^n}|W^n)=0. \tag{29}$$

Next, we suppose that (25) holds. In a manner quite similar to the case of (24), we obtain

$$\liminf_{n\to\infty}\Omega_{n,2,\gamma}(R_2,P_{X^n},P_{Y^n}|W^n)=0. \tag{30}$$

Finally, we assume that (26) holds. Observe that

$$\begin{aligned}&\Omega^{(1)}_{n,3,\gamma}(R_1,R_2,P_{X^n},P_{Y^n}|W^n)\\ &\le \Pr\left\{R_1-\gamma<\frac{1}{n}i_{X^nZ^n}(X^n;Z^n)\right\}\\ &\quad+\Pr\left\{R_2-\gamma<\frac{1}{n}i_{Y^nZ^n}(Y^n;Z^n)\right\}\\ &\quad+\Pr\left\{R_1+R_2-2\gamma<\frac{1}{n}i_{X^nY^nZ^n}(X^nY^n;Z^n)\right\}.\end{aligned} \tag{31}$$

By (26), (31), and the definitions of $\overline{I}(\boldsymbol{X};\boldsymbol{Z})$, $\overline{I}(\boldsymbol{Y};\boldsymbol{Z})$, and $\overline{I}(\boldsymbol{XY};\boldsymbol{Z})$, for any $\gamma\in[0,\gamma_0)$, we have

$$\lim_{n\to\infty}\Omega^{(1)}_{n,3,\gamma}(R_1,R_2,P_{X^n},P_{Y^n}|W^n)=0. \tag{32}$$

We choose τ so that $\tau=(1/2)(\,\gamma+\gamma_0)$. By Property 1 part a), we have

$$\begin{aligned}&\Omega^{(2)}_{n,3,\gamma}(R_1,R_2,P_{X^n},P_{Y^n}|W^n)\\ &\quad\le 3\mathrm{e}^{-n\tau}+\Omega^{(1)}_{n,3,\tau}(R_1,R_2,P_{X^n},P_{Y^n}|W^n)\\ &\qquad-\Omega^{(1)}_{n,3,\gamma}(R_1,R_2,P_{X^n},P_{Y^n}|W^n).\end{aligned} \tag{33}$$

From (32) and (33), for any $\gamma\in[0,\gamma_0)$, we have

$$\lim_{n\to\infty}\Omega_{n,3,\gamma}(R_1,R_2,P_{X^n},P_{X^n}|W^n)=0. \tag{34}$$

From (29), (30), and (34), we have

$$\lim_{n\to\infty}\Omega_{n,\gamma}(R_1,R_2,P_{X^n},P_{Y^n}|W^n)=0$$

for any $\gamma\in[0,\gamma_0)$ and for any $(\boldsymbol{X},\,\boldsymbol{Y})\in\mathcal{S}_I$. Hence, by the definition of $\Omega_{n,\gamma}(R_1,\,R_2|W^n)$ we have for any $\gamma\in[0,\gamma_0)$,

$$\lim_{n\to\infty}\Omega_{n,\gamma}(R_1,R_2|W^n)=0, \tag{35}$$

completing the proof. ∎

B. Proof of Lemma 2

In this appendix we shall prove Lemma 2. We first define several quantities necessary for the proof.

Definition 9 (Partial response(Steinberg [8])): Let (X^n,Y^n) be a pair of two independent random vectors with distribution $(P_{X^n},\,P_{Y^n})$. Let S be a subset of $\mathcal{X}^n\times\,\mathcal{Y}^n\times\,\mathcal{Z}^n$. Define a measure on $\mathcal{Z}^n$ by

$$\begin{aligned}Q_S(\boldsymbol{z})=&\sum_{(\boldsymbol{x},\boldsymbol{y})\in\mathcal{X}^n\times\mathcal{Y}^n}W^n(\boldsymbol{z}|\boldsymbol{x},\boldsymbol{y})P_{X^n}(\boldsymbol{x})P_{Y^n}(\boldsymbol{y})\\ &\times\mathbf{1}_S(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})\end{aligned} \tag{36}$$

We call the measure Q_S *the partial response of* $(P_{X^n},\,P_{Y^n})$ *on* S *through noisy channel* W^n. By definition of the partial response, it is obvious that

$$Q=Q_S+Q_{S^c}\,. \tag{37}$$

Note that Q_S is no longer a probability measure.

Let $S_i, i=1,2,3$ be arbitrary subsets of $\mathcal{X}^n\times\,\mathcal{Y}^n\times\,\mathcal{Z}^n$. For $i=1,2,3$ define

$$\begin{aligned}S_{i,Z}&=\left\{\boldsymbol{z}\in\mathcal{Z}^n:(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})\in S_i\text{ for some }\boldsymbol{x},\boldsymbol{y}\right\},\\ S_{i,ZY}&=\{(\boldsymbol{z},\boldsymbol{y})\in\mathcal{Z}^n\times\mathcal{Y}^n:(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})\in S_i\\ &\qquad\text{for some }\boldsymbol{x}\}\,.\end{aligned}$$

For $\boldsymbol{z}\in S_{i,Z}$ define

$$\begin{aligned}S_{i,XY|Z}(\boldsymbol{z})&=\{(\boldsymbol{x},\boldsymbol{y})\in\mathcal{X}^n\times\mathcal{Y}^n:(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})\in S_i\}\,,\\ S_{i,Y|Z}(\boldsymbol{z})&=\{\boldsymbol{y}\in\mathcal{Y}^n:(\boldsymbol{z},\boldsymbol{y})\in S_{i,ZY}\}\,.\end{aligned}$$

For $(\boldsymbol{z},\boldsymbol{y})\in S_{i,ZY}$ define

$$S_{i,X|ZY}(\boldsymbol{z},\boldsymbol{y})=\{\boldsymbol{x}\in\mathcal{X}^n:(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})\in S_i\}\,.$$

Proof of Lemma 2: The proof consists of three steps.

Step 1 (Random Coding Argument): Let $X_j^n, j\in\mathcal{U}_{M_1}$ be a sequence of independently and identically distributed (i.i.d.) random variables each with distribution $P_{X^n}\in\mathcal{P}(\mathcal{X}^n)$. Each output of the above random variables define a map $\tilde{\varphi}_1:\mathcal{U}_{M_1}\to\mathcal{X}^n$. We use this randomly selected $\tilde{\varphi}_1$ as a transformation map. Define

$$\chi_{\boldsymbol{x}}(\boldsymbol{x}')=\begin{cases}1 & \text{if }\boldsymbol{x}=\boldsymbol{x}'\\ 0 & \text{else}\end{cases}$$

Using the above $\tilde{\varphi}_i$, the input distribution $\tilde{P}_{X^n}=\{\tilde{P}_{X^n}(\boldsymbol{x})\}_{\boldsymbol{x}\in\mathcal{X}^n}$ of $\tilde{\varphi}_1(U_{M_1})$ becomes a random variable, having the form

$$\tilde{P}_{X^n}(\boldsymbol{x})=\tilde{P}_{[X_1^n,X_2^n,\cdots,X_{M_1}^n]}(\boldsymbol{x})=\frac{1}{M_1}\sum_{j=1}^{M_1}\chi_{\boldsymbol{x}}(X_j^n)\,.$$

Similarly, let $Y_j^n, j\in\mathcal{U}_{M_2}$ be a sequence of i.i.d. random variables each with distribution $P_{Y^n}\in\mathcal{P}(\mathcal{Y}^n)$. Each output

of the above random variables define a map $\tilde{\varphi}_2 : \mathcal{U}_{M_2} \to \mathcal{Y}^n$. We use this randomly selected $\tilde{\varphi}_2$ as a transformation map. Using the above $\tilde{\varphi}_2$, the input distribution $\tilde{P}_{Y^n} = \{\tilde{P}_{Y^n}(\boldsymbol{y})\}_{\boldsymbol{y}\in\mathcal{Y}^n}$ of $\tilde{\varphi}_2(U_{M_2})$ becomes a random variable, having the form

$$\tilde{P}_{Y^n}(\boldsymbol{y}) = \tilde{P}_{[Y_1^n, Y_2^n, \cdots, Y_{M_2}^n]}(\boldsymbol{y}) = \frac{1}{M_2}\sum_{j=1}^{M_2} \chi_{\boldsymbol{y}}(Y_j^n)\,.$$

Note that

$$\begin{aligned}
\mathbf{E}\left[\tilde{Q}_{S_1}^{(1)}(\boldsymbol{z})\right] &= \mathbf{E}\left[\tilde{Q}_{S_1[X_1^n, X_2^n, \cdots, X_{M_1}^n]}^{(1)}(\boldsymbol{z})\right] \\
&= Q_{S_1}(\boldsymbol{z}), \qquad (38)\\
\mathbf{E}\left[\tilde{Q}_{S_2}^{(2)}(\boldsymbol{z})\right] &= \mathbf{E}\left[\tilde{Q}_{S_2[Y_1^n, Y_2^n, \cdots, Y_{M_2}^n]}^{(2)}(\boldsymbol{z})\right] \\
&= Q_{S_2}(\boldsymbol{y}), \qquad (39)\\
\mathbf{E}\left[\tilde{Q}_{S_3}^{(3)}(\boldsymbol{z})\right] &= \mathbf{E}\left[\tilde{Q}_{3,S_3[X_1^n Y_1^n, X_2^n Y_2^n, \cdots, X_{M_1}^n Y_{M_2}^n]}(\boldsymbol{z})\right] \\
&= Q_{S_3}(\boldsymbol{z}). \qquad (40)
\end{aligned}$$

Step 2 (Estimation of the Variational Distance): On the upper bound of $d(Q, \tilde{Q}_{i,S_i})$, we obtain the following chain of inequalities:

$$\begin{aligned}
& d(Q, \tilde{Q}^{(i)}) \\
&= \sum_{\boldsymbol{z}\in\mathcal{Z}^n} |\tilde{Q}^{(i)}(\boldsymbol{z}) - Q(\boldsymbol{z})| \\
&= \sum_{\boldsymbol{z}\in\mathcal{Z}^n} |\tilde{Q}_{S_i}^{(i)}(\boldsymbol{z}) + \tilde{Q}_{S_i^c}^{(i)}(\boldsymbol{z}) - Q_{S_i}(\boldsymbol{z}) - Q_{S_i^c}(\boldsymbol{z})| \\
&\le \sum_{\boldsymbol{z}\in\mathcal{Z}^n} \left\{ |\tilde{Q}_{S_i}^{(i)}(\boldsymbol{z}) - Q_{S_i}(\boldsymbol{z})| + \tilde{Q}_{S_i^c}^{(i)}(\boldsymbol{z}) + Q_{S_i^c}(\boldsymbol{z}) \right\} \\
&= \sum_{\boldsymbol{z}\in S_{i,Z}} |\tilde{Q}_{S_i}^{(i)}(\boldsymbol{z}) - Q_{S_i}(\boldsymbol{z})| + \sum_{\boldsymbol{z}\in S_{i,Z}^c} \tilde{Q}_{S_i^c}^{(i)}(\boldsymbol{z}) \qquad (41)\\
&\quad + \mathsf{E}\left[\mathbf{1}_{S^c}(X^n, Y^n, Z^n)\right] .
\end{aligned}$$

Next we evaluate the first and second terms in the right member of (41). For $i = 1, 2, 3$, set

$$\Lambda_i \overset{\triangle}{=} \sum_{\boldsymbol{z}\in S_{i,Z}^c} \tilde{Q}_{S_i^c}^{(i)}(\boldsymbol{z}), \Phi_i \overset{\triangle}{=} \sum_{\boldsymbol{z}\in S_{i,Z}} |\tilde{Q}_{S_i}^{(i)}(\boldsymbol{z}) - Q_{S_i}(\boldsymbol{z})|\,.$$

We first observe that

$$\mathbf{E}[\Lambda_i] = \mathsf{E}\left[\mathbf{1}_{S_i^c}(X^n, Y^n, Z^n)\right],\ i = 1, 2, 3. \qquad (42)$$

Next we derive upper bounds of $\Phi_i, i = 1, 2, 3$. We first derive an upper bound of Φ_1. Observe that

$$\begin{aligned}
& \Phi_1 \\
&\le \sum_{\boldsymbol{y}\in S_{1,Y}} P_{Y^n}(\boldsymbol{y}) \\
&\quad \times \sum_{\boldsymbol{z}\in S_{1,Z|Y}(\boldsymbol{y})} |\tilde{P}_{Z^n|Y^n,S_1}^{(1)}(\boldsymbol{z}|\boldsymbol{y}) - P_{Z^n|Y^n,S_1}(\boldsymbol{z}|\boldsymbol{y})|\,.
\end{aligned}$$

Applying the Cauchy-Schwartz inequality and using the concavity of $\sqrt{x}$, we have

$$\begin{aligned}
& \Phi_1 \\
&\le \sum_{\boldsymbol{y}\in S_{1,Y}} P_{Y^n}(\boldsymbol{y}) \times \left\{P_{Z^n|Y^n}(S_{1,Z|Y}(\boldsymbol{y})|\boldsymbol{y})\right\}^{1/2} \\
&\quad \times \left\{ \sum_{\boldsymbol{z}\in S_{1,Z|Y}(\boldsymbol{y})} \frac{\left\{\tilde{P}_{Z^n|Y^n,S_1}^{(1)}(\boldsymbol{z}|\boldsymbol{y}) - P_{Z^n|Y^n,S_1}(\boldsymbol{z}|\boldsymbol{y})\right\}^2}{P_{Z^n|Y^n}(\boldsymbol{z}|\boldsymbol{y})} \right\}^{1/2} \\
&\le \sum_{\boldsymbol{y}\in S_{1,Y}} P_{Y^n}(\boldsymbol{y}) \\
&\quad \times \left\{ \sum_{\boldsymbol{z}\in S_{1,Z|Y}(\boldsymbol{y})} \frac{\left\{\tilde{P}_{Z^n|Y^n,S_1}^{(1)}(\boldsymbol{z}|\boldsymbol{y}) - P_{Z^n|Y^n,S_1}(\boldsymbol{z}|\boldsymbol{y})\right\}^2}{P_{Z^n|Y^n}(\boldsymbol{z}|\boldsymbol{y})} \right\}^{1/2} \\
&\le \left\{P_{Y^n}(S_{1,Y})\right\}^{1/2} \\
&\quad \times \left\{ \sum_{\boldsymbol{y}\in S_{1,Y}} P_{Y^n}(\boldsymbol{y}) \right. \\
&\quad \left. \times \sum_{\boldsymbol{z}\in S_{1,Z|Y}(\boldsymbol{y})} \frac{\left\{\tilde{P}_{Z^n|Y^n,S_1}^{(1)}(\boldsymbol{z}|\boldsymbol{y}) - P_{Z^n|Y^n,S_1}(\boldsymbol{z}|\boldsymbol{y})\right\}^2}{P_{Z^n|Y^n}(\boldsymbol{z}|\boldsymbol{y})} \right\}^{1/2} \\
&\le \left\{ \sum_{(\boldsymbol{z},\boldsymbol{y})\in S_{1,ZY}} P_{Y^n}(\boldsymbol{y}) \right. \\
&\quad \left. \times \frac{\left\{\tilde{P}_{Z^n|Y^n,S_1}^{(1)}(\boldsymbol{z}|\boldsymbol{y}) - P_{Z^n|Y^n,S_1}(\boldsymbol{z}|\boldsymbol{y})\right\}^2}{P_{Z^n|Y^n}(\boldsymbol{z}|\boldsymbol{y})} \right\}^{1/2} \qquad (43)
\end{aligned}$$

Taking expectation of both sides of (43) and using Jensen's inequality, we have

$$\begin{aligned}
& \mathbf{E}[\Phi_1] \\
&\le \left\{ \sum_{(\boldsymbol{z},\boldsymbol{y})\in S_{1,ZY}} P_{Y^n}(\boldsymbol{y}) \frac{\mathbf{Var}\left[\tilde{P}_{Z|Y,S_1}^{(1)}(\boldsymbol{z}|\boldsymbol{y})\right]}{P_{Z^n|Y^n}(\boldsymbol{z}|\boldsymbol{y})} \right\}^{1/2} .
\end{aligned}$$

In a manner quite similar to the above argument we obtain

$$\begin{aligned}
& \mathbf{E}[\Phi_2] \\
&\le \left\{ \sum_{(\boldsymbol{z},\boldsymbol{x})\in S_{2,ZX}} P_{X^n}(\boldsymbol{x}) \frac{\mathbf{Var}\left[\tilde{P}_{Z|X,S_2}^{(2)}(\boldsymbol{z}|\boldsymbol{x})\right]}{P_{Z^n|X^n}(\boldsymbol{z}|\boldsymbol{x})} \right\}^{1/2} .
\end{aligned}$$

Next, we derive an upper bound of Φ_3. Applying the Cauchy-Schwartz inequality, we have

$$\Phi_3 \le \left\{ \sum_{\boldsymbol{z}\in S_{3,Z}} Q(\boldsymbol{z}) \right\}^{1/2}$$

$$\times \left\{ \sum_{\boldsymbol{z}\in S_{3,Z}} \frac{\left\{\tilde{Q}^{(3)}_{S_3}(\boldsymbol{z})-Q_{S_3}(\boldsymbol{z})\right\}^2}{Q(\boldsymbol{z})} \right\}^{1/2}$$

$$\leq \left\{ \sum_{\boldsymbol{z}\in S_{3,Z}} \frac{\left\{\tilde{Q}^{(3)}_{S_3}(\boldsymbol{z})-Q_{S_3}(\boldsymbol{z})\right\}^2}{Q(\boldsymbol{z})} \right\}^{1/2} \tag{44}$$

Taking expectation of both sides of (44) and using Jensen's inequality, we have

$$\mathbf{E}[\Phi_3] \leq \left\{ \sum_{\boldsymbol{z}\in S_{3,Z}} \frac{\mathbf{Var}\left[\tilde{Q}^{(3)}_{S_3}(\boldsymbol{z})\right]}{Q(\boldsymbol{z})} \right\}^{1/2}. \tag{45}$$

Step 3(Computation of the Variances) : Observe that

$$\begin{aligned}
&\left\{\tilde{P}^{(1)}_{Z^n|Y^n,S_1}(\boldsymbol{z}|\boldsymbol{y})\right\}^2\\
&=\frac{1}{M_1^2}\sum_{j=1}^{M_1}\sum_{\boldsymbol{x}\in S_{1,X|ZY}(\boldsymbol{z},\boldsymbol{y})}\left[W^n(\boldsymbol{z}|\boldsymbol{x},\boldsymbol{y})\right]^2\chi_{\boldsymbol{x}}(X_j^n)\\
&\quad+\frac{1}{M_1^2}\sum_{j\neq j'}\sum_{\boldsymbol{x}\in S_{1,X|ZY}(\boldsymbol{z},\boldsymbol{y})}\sum_{\boldsymbol{x}'\in S_{1,X|ZY}(\boldsymbol{z},\boldsymbol{y})}\\
&\qquad\times W^n(\boldsymbol{z}|\boldsymbol{x},\boldsymbol{y})W^n(\boldsymbol{z}|\boldsymbol{x}',\boldsymbol{y})\chi_{\boldsymbol{x}}(X_j^n)\chi_{\boldsymbol{x}'}(X_{j'}^n).
\end{aligned} \tag{46}$$

Taking expectation of both sides of (46), we obtain

$$\begin{aligned}
&\mathbf{E}\left[\left\{\tilde{P}^{(1)}_{Z^n|Y^n,S_1}(\boldsymbol{z}|\boldsymbol{y})\right\}^2\right]\\
&\leq\frac{1}{M_1}\sum_{\boldsymbol{x}\in S_{1,X|ZY}(\boldsymbol{z},\boldsymbol{y})}\left[W^n(\boldsymbol{z}|\boldsymbol{x},\boldsymbol{y})\right]^2P_{X^n}(\boldsymbol{x})\\
&\quad+\left\{P_{Z^n|Y^n,S_1}(\boldsymbol{z}|\boldsymbol{y})\right\}^2.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\mathbf{Var}\left[\tilde{P}^{(1)}_{Z^n|Y^n,S_1}(\boldsymbol{z}|\boldsymbol{y})\right]\\
&\leq\frac{1}{M_1}\sum_{\boldsymbol{x}\in S_{1,X|ZY,\gamma}(\boldsymbol{z},\boldsymbol{y})}\left[W^n(\boldsymbol{z}|\boldsymbol{x},\boldsymbol{y})\right]^2P_{X^n}(\boldsymbol{x}).
\end{aligned}$$

From the above inequality and (44), we obtain

$$\begin{aligned}
&\mathbf{E}\left[\Phi_1\right]\\
&\leq\left\{\sum_{(\boldsymbol{z},\boldsymbol{y})\in S_{1,ZY}}\sum_{\boldsymbol{x}\in S_{1,X|ZY}(\boldsymbol{z},\boldsymbol{y})}P_{Y^n}(\boldsymbol{y})\frac{[W^n(\boldsymbol{z}|\boldsymbol{x},\boldsymbol{y})]^2P_{X^n}(\boldsymbol{x})}{M_1P_{Z^n|Y^n}(\boldsymbol{z}|\boldsymbol{y})}\right\}^{1/2}\\
&=\left\{\sum_{(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})\in S_1}\exp\left\{-n\left[R_1-\tfrac{1}{n}i_{X^nY^nZ^n}(\boldsymbol{x};\boldsymbol{z}|\boldsymbol{y})\right]\right\}\right.\\
&\qquad\left.\times W^n(\boldsymbol{z}|\boldsymbol{x},\boldsymbol{y})P_{X^n}(\boldsymbol{x})P_{Y^n}(\boldsymbol{y})\right\}^{1/2}\\
&=\sqrt{\zeta_{n,1,S_1}(R_1,P_{X^n},P_{Y^n}|W^n)}.
\end{aligned} \tag{47}$$

In a manner quite similar to the above argument we obtain

$$\mathbf{E}\left[\Phi_2\right]\leq\sqrt{\zeta_{n,2,S_2}(R_2,P_{X^n},P_{Y^n}|W^n)}. \tag{48}$$

Next, we compute $\mathbf{Var}[\tilde{Q}^{(3)}_{S_3}(\boldsymbol{z})]$. Observe that

$$\begin{aligned}
&\left\{\tilde{Q}^{(3)}_{S_3}(\boldsymbol{z})\right\}^2\\
&=\frac{1}{M_1^2M_2^2}\sum_{j=1}^{M_1}\sum_{k=1}^{M_2}\sum_{(\boldsymbol{x},\boldsymbol{y})\in S_{3,XY|Z}(\boldsymbol{z})}\\
&\quad\times\left[W^n(\boldsymbol{y}|\boldsymbol{x},\boldsymbol{y})\right]^2\chi_{\boldsymbol{x}}(X_j^n)\chi_{\boldsymbol{y}}(Y_k^n)\\
&\quad+\frac{1}{M_1^2M_2^2}\sum_{j\neq j'}\sum_{k=1}^{M_2}\sum_{\substack{(\boldsymbol{x},\boldsymbol{y})\in S_{3,XY|Z}(\boldsymbol{z})\\(\boldsymbol{x}',\boldsymbol{y})\in S_{3,XY|Z}(\boldsymbol{z})}}\\
&\quad\times W^n(\boldsymbol{z}|\boldsymbol{x},\boldsymbol{y})W^n(\boldsymbol{z}|\boldsymbol{x}',\boldsymbol{y})\chi_{\boldsymbol{x}}(X_j^n)\chi_{\boldsymbol{x}'}(X_{j'}^n)\chi_{\boldsymbol{y}}(Y_k^n)\\
&\quad+\frac{1}{M_1^2M_2^2}\sum_{j=1}^{M_1}\sum_{k\neq k'}\sum_{\substack{(\boldsymbol{x},\boldsymbol{y})\in S_{3,XY|Z}(\boldsymbol{z})\\(\boldsymbol{x},\boldsymbol{y}')\in S_{3,XY|Z}(\boldsymbol{z})}}\\
&\quad\times W^n(\boldsymbol{z}|\boldsymbol{x},\boldsymbol{y})W^n(\boldsymbol{z}|\boldsymbol{x},\boldsymbol{y}')\chi_{\boldsymbol{x}}(X_j^n)\chi_{\boldsymbol{y}}(Y_k^n)\chi_{\boldsymbol{y}'}(Y_{k'}^n)\\
&\quad+\frac{1}{M_1^2M_2^2}\sum_{j\neq j'}\sum_{k\neq k'}\sum_{\substack{(\boldsymbol{x},\boldsymbol{y})\in S_{3,XY|Z}(\boldsymbol{z})\\(\boldsymbol{x}',\boldsymbol{y}')\in S_{3,XY|Z}(\boldsymbol{z})}}\\
&\quad\times W^n(\boldsymbol{z}|\boldsymbol{x},\boldsymbol{y})W^n(\boldsymbol{z}|\boldsymbol{x}',\boldsymbol{y}')\\
&\quad\times\chi_{\boldsymbol{x}}(X_j^n)\chi_{\boldsymbol{x}'}(X_{j'}^n)\chi_{\boldsymbol{y}}(Y_k^n)\chi_{\boldsymbol{y}'}(Y_{k'}^n).
\end{aligned} \tag{49}$$

Taking expectation of both sides of (49), we obtain

$$\begin{aligned}
&\mathbf{E}\left[\left\{\tilde{Q}^{(3)}_{S_3}(\boldsymbol{z})\right\}^2\right]\\
&\leq\frac{1}{M_1M_2}\sum_{(\boldsymbol{x},\boldsymbol{y})\in S_{3,XY|Z}(\boldsymbol{z})}\left[W^n(\boldsymbol{z}|\boldsymbol{x},\boldsymbol{y})\right]^2P_{X^n}(\boldsymbol{x})P_{Y^n}(\boldsymbol{y})\\
&\quad+\frac{1}{M_2}\sum_{\boldsymbol{y}\in S_{3,Y|Z}(\boldsymbol{z})}\sum_{\boldsymbol{x},\boldsymbol{x}'\in S_{3,X|YZ}(\boldsymbol{z},\boldsymbol{y})}\\
&\quad\times W^n(\boldsymbol{z}|\boldsymbol{x},\boldsymbol{y})W^n(\boldsymbol{z}|\boldsymbol{x}',\boldsymbol{y})P_{X^n}(\boldsymbol{x})P_{X^n}(\boldsymbol{x}')P_{Y^n}(\boldsymbol{y})\\
&\quad+\frac{1}{M_1}\sum_{\boldsymbol{x}\in S_{3,X|Z}(\boldsymbol{z})}\sum_{\boldsymbol{y},\boldsymbol{y}'\in S_{3,Y|ZX}(\boldsymbol{z},\boldsymbol{x})}\\
&\quad\times W^n(\boldsymbol{z}|\boldsymbol{x},\boldsymbol{y})W^n(\boldsymbol{z}|\boldsymbol{x},\boldsymbol{y}')P_{X^n}(\boldsymbol{x})P_{Y^n}(\boldsymbol{y})P_{Y^n}(\boldsymbol{y}')\\
&\quad+\left\{Q_{S_3}(\boldsymbol{z})\right\}^2.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\mathbf{Var}\left[\tilde{Q}^{(3)}_{S_3}(\boldsymbol{z})\right]\\
&\leq\frac{1}{M_1M_2}\sum_{(\boldsymbol{x},\boldsymbol{y})\in S_{3,XY|Z}(\boldsymbol{z})}\left[W^n(\boldsymbol{z}|\boldsymbol{x},\boldsymbol{y})\right]^2P_{X^n}(\boldsymbol{x})P_{Y^n}(\boldsymbol{y})\\
&\quad+\frac{1}{M_2}\sum_{\boldsymbol{y}\in S_{3,Y|Z}(\boldsymbol{z})}\left[W^n(\boldsymbol{z}|\boldsymbol{y})\right]^2P_{Y^n}(\boldsymbol{y})\\
&\quad+\frac{1}{M_1}\sum_{\boldsymbol{x}\in S_{3,X|Z}(\boldsymbol{z})}\left[W^n(\boldsymbol{z}|\boldsymbol{x})\right]^2P_{X^n}(\boldsymbol{x}).
\end{aligned}$$

From the above inequality and (45), we obtain

$$\mathbf{E}\left[\Phi_3\right]$$

$$
\begin{aligned}
&\le \Biggl\{ \sum_{\boldsymbol{z}\in S_{3,Z}} \sum_{(\boldsymbol{x},\boldsymbol{y})\in S_{3,XY|Z}(\boldsymbol{z})} \frac{\left[W^n(\boldsymbol{z}|\boldsymbol{x},\boldsymbol{y})\right]^2 P_{X^n}(\boldsymbol{x})P_{Y^n}(\boldsymbol{y})}{M_1M_2Q(\boldsymbol{z})} \\
&\quad + \sum_{\boldsymbol{z}\in S_{3,Z}} \sum_{\boldsymbol{y}\in S_{3,Y|Z}(\boldsymbol{z})} \frac{\left[W^n(\boldsymbol{z}|\boldsymbol{y})\right]^2 P_{Y^n}(\boldsymbol{y})}{M_2Q(\boldsymbol{z})} \\
&\quad + \sum_{\boldsymbol{z}\in S_{3,Z}} \sum_{\boldsymbol{x}\in S_{3,X|Z}(\boldsymbol{z})} \frac{\left[W^n(\boldsymbol{z}|\boldsymbol{x})\right]^2 P_{X^n}(\boldsymbol{x})}{M_1Q(\boldsymbol{z})} \Biggr\}^{1/2} \\
&= \Biggl\{ \sum_{(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})\in S_3} \exp\left\{-n\left[R_1+R_2-\tfrac{1}{n}i_{X^nY^nZ^n}(\boldsymbol{x}\boldsymbol{y};\boldsymbol{z})\right]\right\} \\
&\qquad\qquad \times W^n(\boldsymbol{z}|\boldsymbol{x}\boldsymbol{y})P_{X^n}(\boldsymbol{x})P_{Y^n}(\boldsymbol{y}) \\
&\quad + \sum_{(\boldsymbol{y},\boldsymbol{z})\in S_{2,YZ}} \exp\left\{-n\left[R_2-\tfrac{1}{n}i_{Y^nZ^n}(\boldsymbol{y};\boldsymbol{z})\right]\right\} \\
&\qquad\qquad \times W^n(\boldsymbol{z}|\boldsymbol{y})P_{Y^n}(\boldsymbol{y}) \\
&\quad + \sum_{(\boldsymbol{x},\boldsymbol{z})\in S_{1,XZ}} \exp\left\{-n\left[R_1-\tfrac{1}{n}i_{X^nZ^n}(\boldsymbol{x};\boldsymbol{z})\right]\right\} \\
&\qquad\qquad \times W^n(\boldsymbol{z}|\boldsymbol{x})P_{X^n}(\boldsymbol{x}) \Biggr\}^{1/2} \\
&= \sqrt{\zeta_{n,3,S_3}(R_1,R_2,P_{X^n},P_{Y^n}|W^n)}\,. \qquad (50)
\end{aligned}
$$

Set

$$\Theta_i \stackrel{\triangle}{=} \mathsf{E}\left[\mathbf{1}_{S_i^c}(X^n,Y^n,Z^n)\right] + \sqrt{\zeta_{n,i,S_i}},\ i=1,2,3.$$

From (42), (47), (48), and (50), we obtain

$$
\begin{aligned}
&\mathbf{E}\left[\sum_{i=1,2,3} \Theta_i^{-1}(\Lambda_i+\Phi_i)\right] \\
&= \sum_{i=1,2,3} \Theta_i^{-1}\left\{\mathbf{E}[\Lambda_i]+\mathbf{E}[\Phi_i]\right\} \\
&\le \sum_{i=1,2,3} \Theta_i^{-1}\left\{\mathsf{E}\left[\mathbf{1}_{S_i^c}(X^n,Y^n,Z^n)\right]+\sqrt{\zeta_{n,i,S_i}}\right\} = 3.
\end{aligned}
$$

Then, there exists at least one deterministic maps $\tilde{\varphi}_i, i=1,2$ such that

$$\sum_{i=1,2,3} \Theta_i^{-1}(\Lambda_i+\Phi_i) \le 3,$$

from which we have

$$\Lambda_i+\Phi_i \le 3\Theta_i, i=1,2,3. \qquad (51)$$

From (41) and (51), we obtain

$$
\begin{aligned}
&d(Q,\tilde{Q}^{(i)}) \\
&\le 4\mathsf{E}\left[\mathbf{1}_{S_i^c}(X^n,Y^n,Z^n)\right] + 3\sqrt{\zeta_{n,i,S_i}},\ i=1,2,3,
\end{aligned}
$$

completing the proof of Lemma 2 . ■

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